Nonlinear, Inelastic ESSI Analysis

SMiRT26 Tutorial I

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Outline

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Motivation

Improve modeling and simulation for infrastructure objects

Reduction of modeling, epistemic uncertainty

Propagate parametric, aleatory uncertainty

Predict and inform, Engineer needs to know!

Design, build and maintain sustainable objects

High consequence, important objects

Analysis must be better than "rocket science"





Dedication

Robert P. Kennedy, 1939-2018

"Response of a soil structure system is nonlinear, and I would really like to know what that response is!"

Nebojša Orbović, 1962-2021

"As an engineer, I have to know what are response sensitivities to modeling choices and model parameters."

Hypothesis

- Interplay of the Earthquake, Soil/Rock and Structure in time domain, plays a major role in successes and failures
- Timing and spatial location of energy dissipation determines location and amount of damage
- If timing and spatial location of the energy dissipation can be controlled (directed), we could optimize soil structure system for
  - Safety
  - Economy

ESSI: Energy Input and Dissipation

Energy input, dynamic forcing

Energy dissipation outside SSI domain:

- SSI system oscillation radiation
- Reflected wave radiation

Energy dissipation/conversion inside SSI domain:

- Inelasticity of soil, contact/interface zone, structure, foundation, dissipators
- Viscous coupling, porous solid-pore fluids, solids/structures-external fluids

Numerical, algorithmic energy dissipation/production

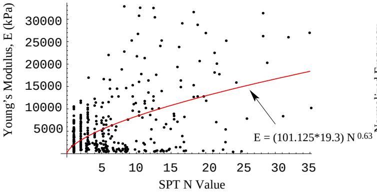
Prediction under Uncertainty

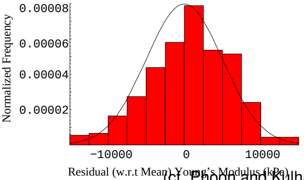
- Modeling, Epistemic Uncertainties
  - Modeling simplifications
  - Low, medium, high sophistication modeling and simulation
  - Modeling sophistication level for confidence in results
  - Verification and Validation
- Parametric, Aleatory Uncertainties
  - $M\ddot{u}_i + C\dot{u}_i + K^{ep}u_i = F(t)$
  - Uncertain: mass M, viscous damping C and stiffness K^{ep}
  - Uncertain loads, $F(t)$
  - Results are PDFs and CDFs for σ_{ij}, ϵ_{ij}, u_i, $\dot{u}_i$, $\ddot{u}_i$

Modeling, Epistemic Uncertainty

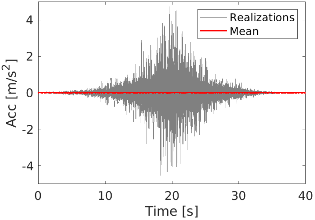
- Important (?!) features are simplified
  - 1C vs 3C seismic motions
  - Elastic vs Inelastic behavior
  - Interaction with fluid(s)
- Modeling simplifications are justifiable if one or two level higher sophistication model demonstrates that features being simplified out are not important

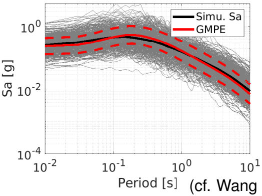
Parametric, Aleatory Uncertainty





(cf. Phoon and Kulhawy (1999B))





(cf. Wang et al. (2019))

Outline

Analysis Governance

- Numerical analysis is fragile
- Engineer's competency and expertise
- Model verification
- Solution verification, program-mathematics inaccuracies
- Validation, program-physics inaccuracies

Engineer, Analyst

- Sound engineering judgement
- Assess various analysis sophistication levels
- Engineer is in full control of the model and the analysis
- Engineer uses models to investigate designs
- Confidence in all modeling choices
- Confidence in all model components
- Confidence in all analysis results

Analysis Program

- Hierarchy of model sophistication
- Hierarchy of simulation/algorithmic capabilities
- Full (!) Verification
- Extensive Validation
- Confidence in analysis results

Under the Simulation Hood

- Commercial programs benchmark examples
- Commercial programs verification (?)
- Surprises under the simulation hood


Commercial Programs

Oberkampf and Trucano, SAND2007-0853 note:

- Commercial programs with large number of benchmark examples
- Primary goal is to demonstrate "engineering accuracy"
- However (!) verification should carefully quantify the numerical error in the solutions

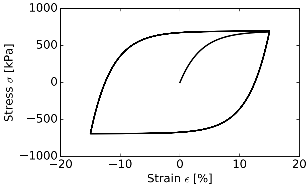
Outline

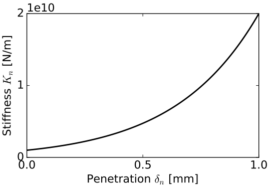
Inelastic Modeling of Soil Structure Systems

- Soil, inelastic, elastic-plastic
  - Dry, single phase
  - Unsaturated, partially saturated
  - Fully saturated
- Contact/interface/joint, inelastic, soil/rock – foundation
  - Dry, single phase,
    - Normal, hard and soft, gap open/close
    - Friction, nonlinear
  - Fully saturated, suction, excess pressure, buoyant force
- Structure, inelastic, damage, cracks
  - Nonlinear/inelastic 1D reinforced concrete fiber beam
  - Nonlinear/inelastic 3D reinforced concrete solid element
  - Alcali Silica Reaction concrete modeling

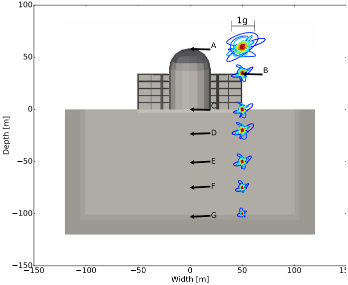
Inelastic Soil and Inelastic Contact/Interface/Joint

- Shear velocity of soil $V_s = 500 m/s$
- Undrained shear strength (Dickenson 1994) $V_s[m/s] = 23(S_u[kPa])^{0.475}$
- For $V_s = 500 m/s$ Undrained Strength $S_u = 650 kPa$ and Young's Modulus of $E = 1.3 GPa$
- von Mises, Armstrong Frederick kinematic hardening ($S_u = 650 kPa$ at $\gamma = 0.01\%$; $h_a = 30 MPa$, $c_r = 25$)
- Soft contact (concrete-soil), gaping and nonlinear shear

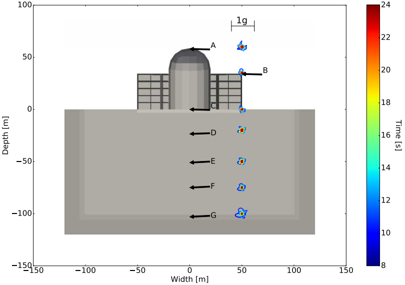




Acceleration Traces, Elastic vs Inelastic

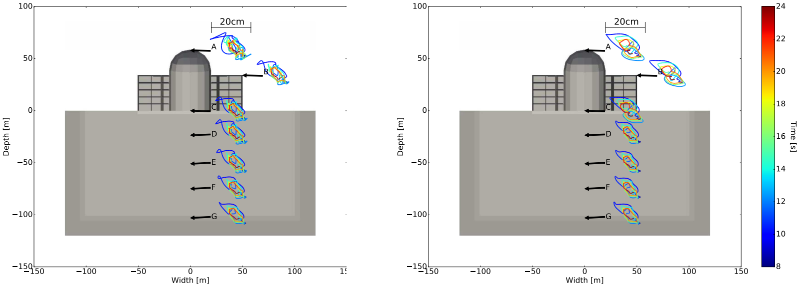


Elastic



Inelastic

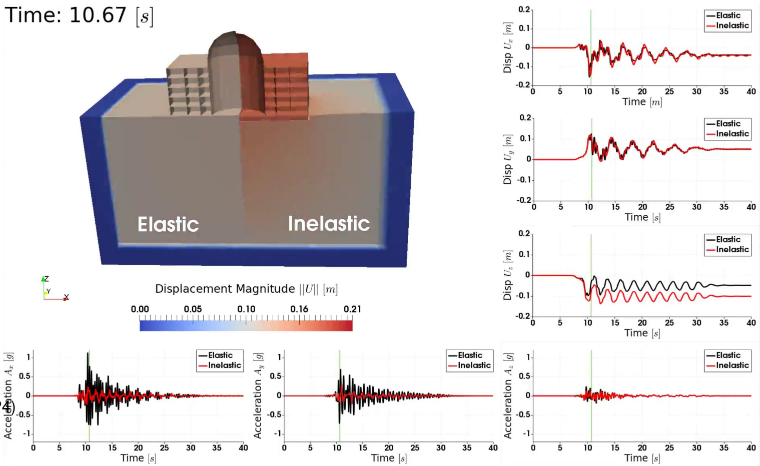
Displacement Traces, Elastic vs Inelastic



Elastic

Inelastic

Elastic and Inelastic Response: Differences



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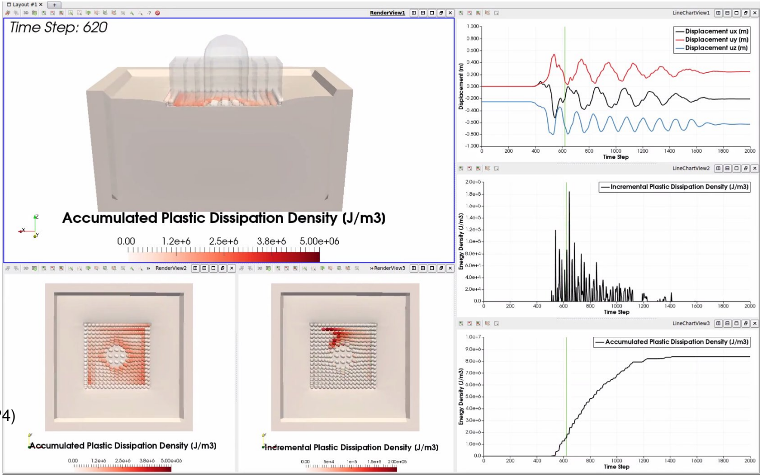
Energy Dissipation in a Large-Scale Model



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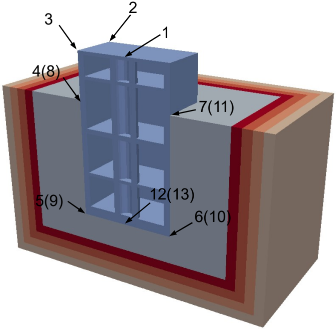
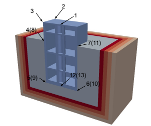
Deeply Embedded Structures

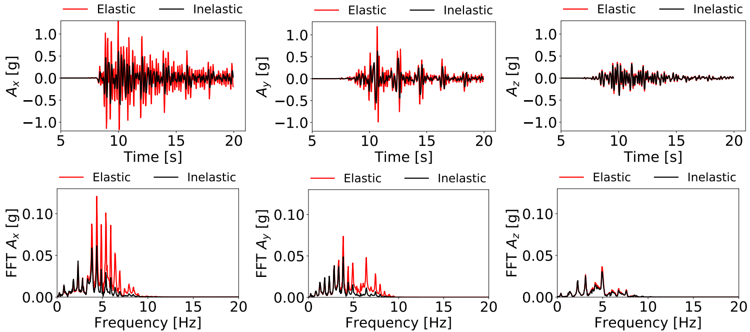


Location of points

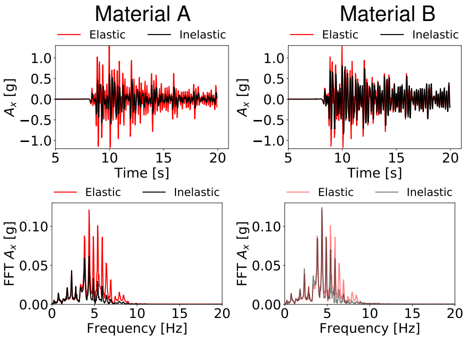
Point ID	X (m)	Y (m)	Z (m)	layer
1	0	0	14	structure
2	15	15	14	structure
3	0	15	14	structure
4	0	15	0	structure
5	0	15	-36	structure
6	0	-15	-36	structure
7	0	-15	0	structure
8	0	15	0	surrounding soil
9	0	15	-36	surrounding soil
10	0	-15	-36	surrounding soil
11	0	-15	0	surrounding soil
12	0	0	-36	structure
13	0	0	-36	surrounding soil

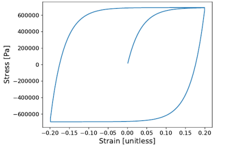


SMR: Inelastic ESSI Effects, Top Center

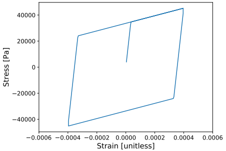


SMR: ESSI Effects, Material Modeling



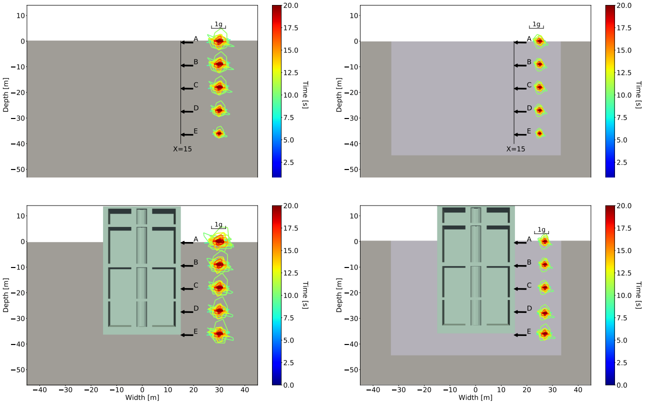


Material A: nonlinear, vM - AF



Material B: Bilinear

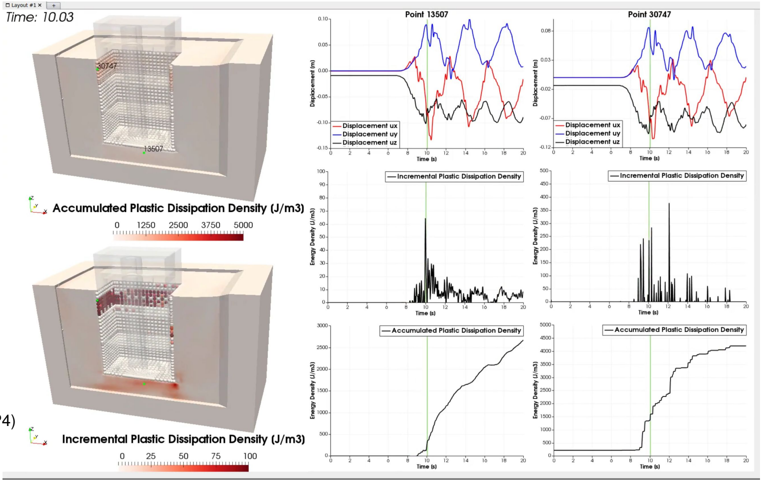
SMR: Accelerations Along Depth



Nonlinear site effects

SSI effects

Energy Dissipation for an SMR Model



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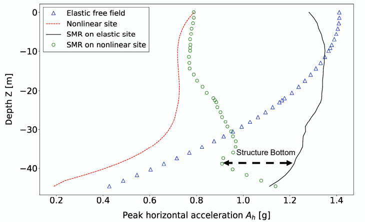
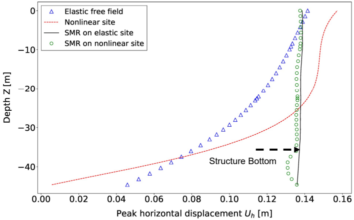
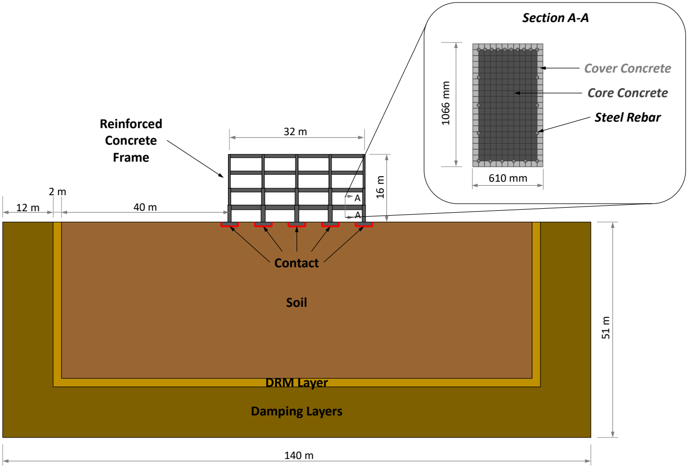
Depth variation - PGA & PGD



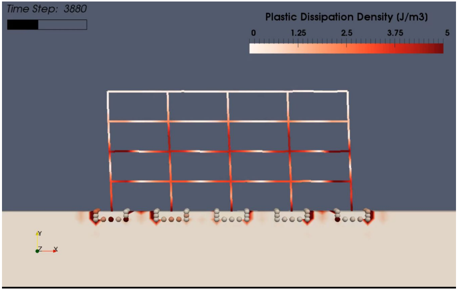


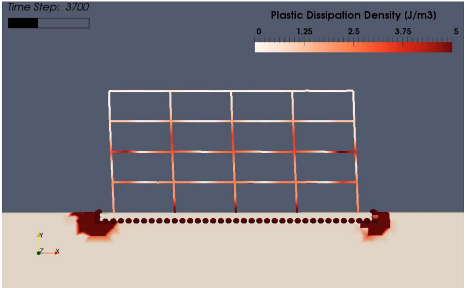
- The PGA & PGD of SSI systems are (very) different from free field motions,
  - Material nonlinearity has significant effect on acceleration response.

Energy Dissipation for Design



Design Alternatives





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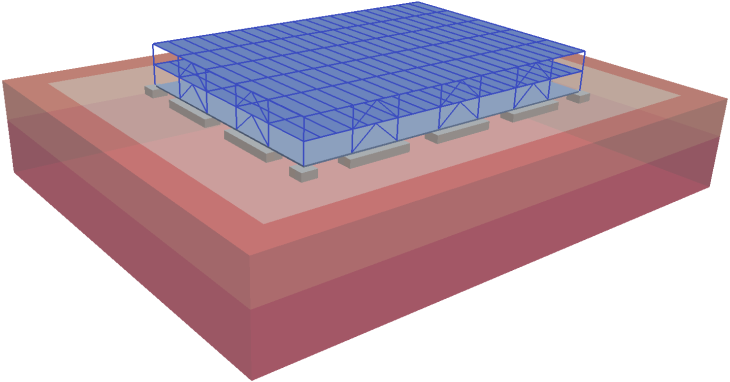
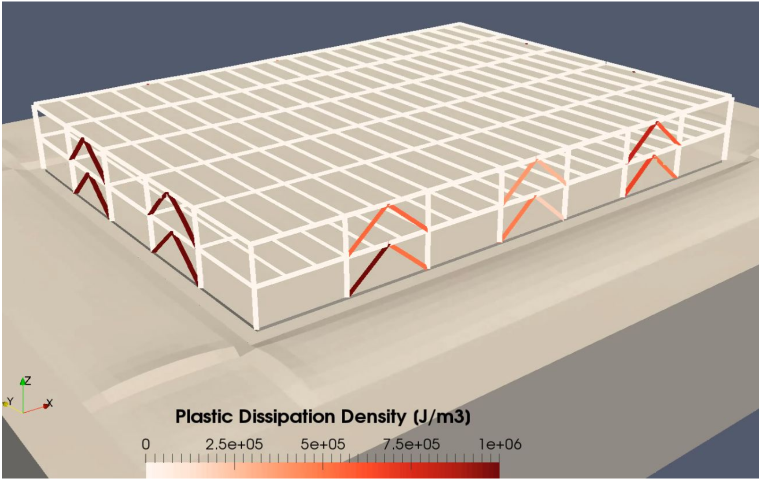
ASCE-7-21: Buildings and Models, Low Building



ASCE-7-21: Low Building Energy Dissipation



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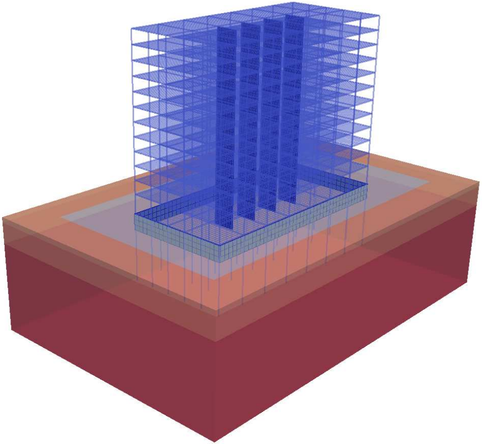
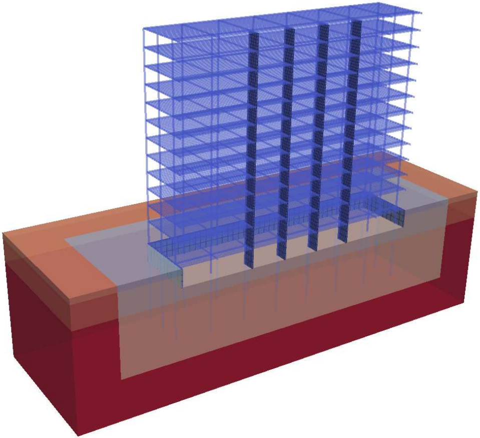
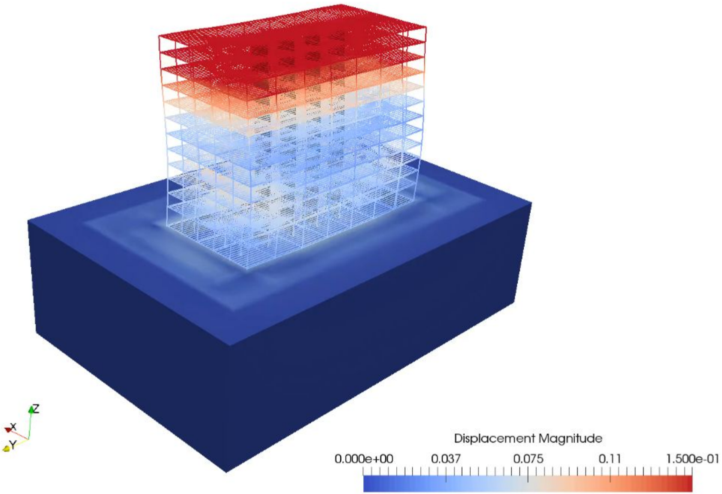
ASCE-7-21: Tall Building



ASCE-7-21: Buildings and Models, Tall Building



ASCE-7-21: Tall Building Response



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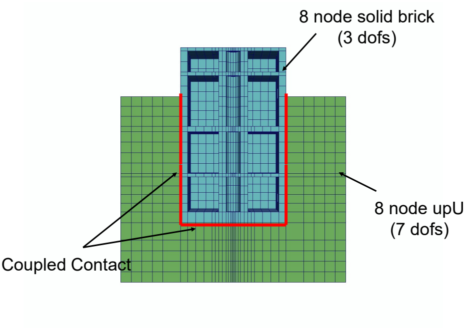
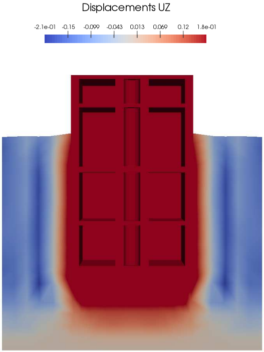
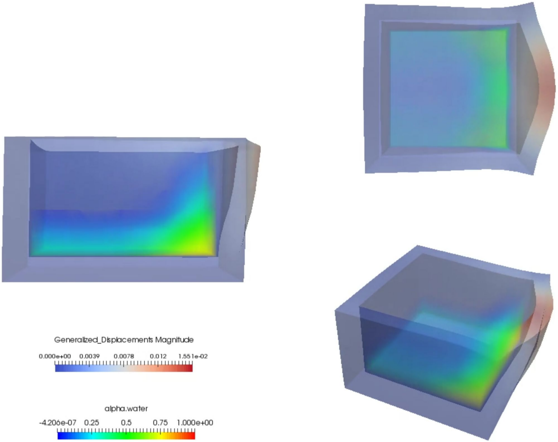
Buoyant Force Simulation





Solid, Structure-Fluid Interaction, Example



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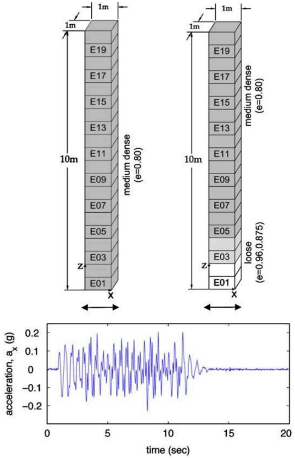
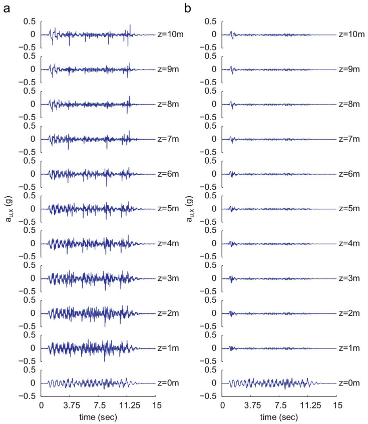
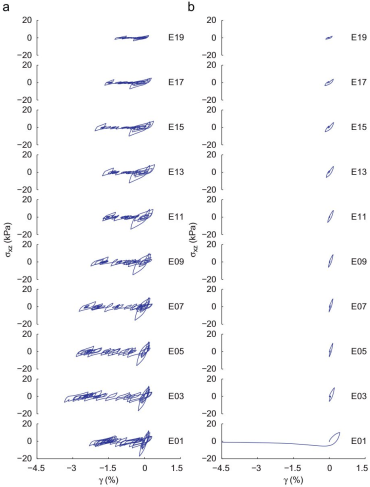
Liquefaction as Base Isolation, Model



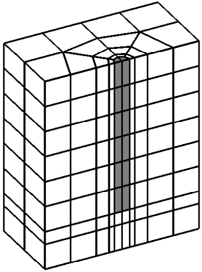
Liquefaction, Wave Propagation

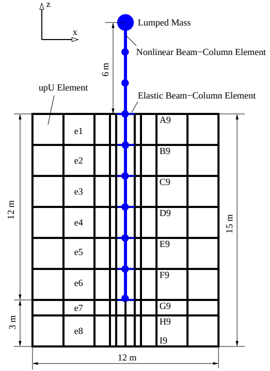


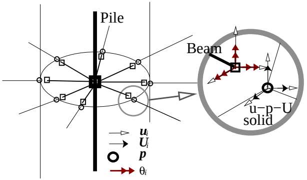
Liquefaction, Stress-Strain Response



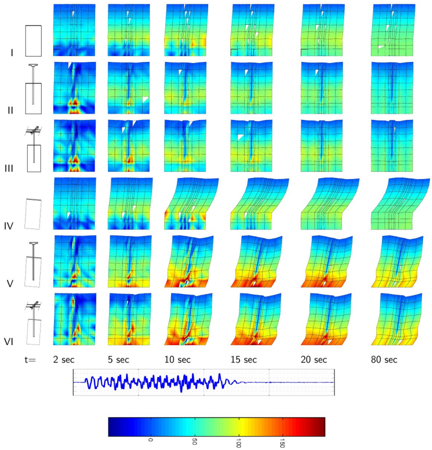
Pile in Liquefiable Soil, Model





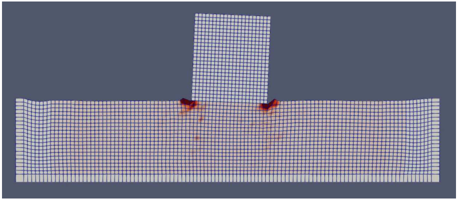


Pile in Liquefiable Soil, Results

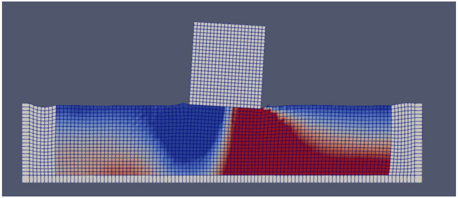


Building on Liquefiable Soil

Plastic Strain



Pore Fluid Pressures



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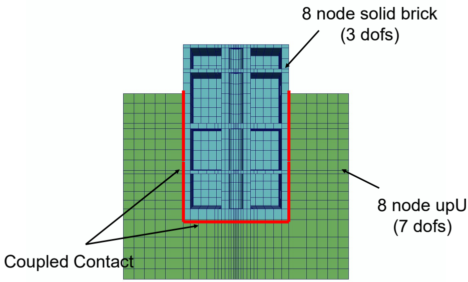
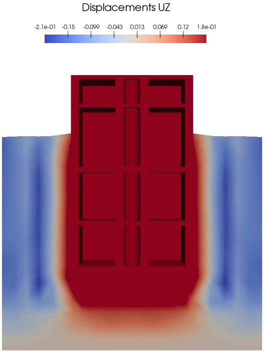
Buoyant Force Simulation





Outline

Analysis Program Inaccuracies

- Numerical approximations
- Analysis Program quality control/assurance/management
- Numerical programs without quality control are dangerous!
- Fitting a curve, getting a number once, does not mean that results are accurate
  (2.0+2.0=4.0; 1.9+2.1=4.0; 2.51+1.51=4.02)

The T Experiments

- Les Hatton: The T experiments, Errors in scientific software. IEEE Computational Science and Engineering, 4(2):27-38, April-June 1997.
- "Extensive tests showed that many software codes widely used in science and engineering are not as accurate as we would like to think."
- "Better software engineering practices wold help solve this problem,"
- "Realizing that the problem exists is an important first step."
- Large experiment over 4 years measuring faults (T1) and failures (T2) of scientific and engineering codes

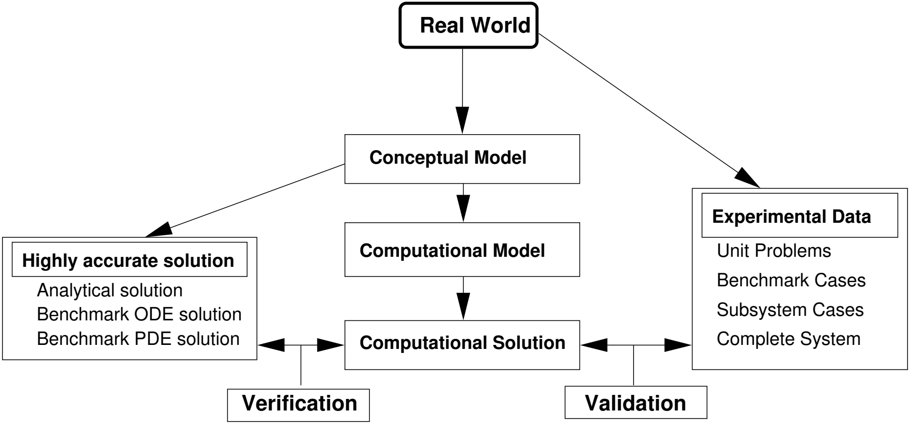
Verification and Validation

- Verification: provides evidence that the model is solved correctly. Mathematics issue. Well developed for the Real ESSI Simulator.
- Validation: provides evidence that the correct model is solved. Physics issue. Work in progress.
- Prediction: use of computational model to foretell the state of a physical system under consideration under conditions for which the computational model has not been validated.

V & V Motivation

- How much can (should) we trust model implementations (verification)?
- How much can (should) we trust numerical simulations (validation)?
- How good are our numerical predictions?
- Can simulation tools be used for improving safety and economy?
- V & V procedures are the primary means of assessing accuracy in modeling and computational simulations
- V & V procedures are the tools with which we build confidence and credibility in modeling and computational simulations

Fundamentals of Verification and Validation



Important Sources

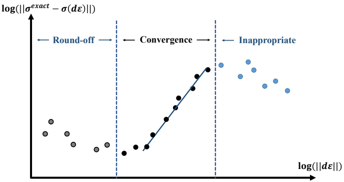
- W. L. OBERKAMPF, T. G. TRUCANO, AND C. HIRSCH. Verification, validation and predictive capability in computational engineering and physics. In Proceedings of the Foundations for Verification and Validation on the 21st Century Workshop, pages 1–74, Laurel, Maryland, October 22-23 2002. Johns Hopkins University / Applied Physics Laboratory.
- P. J. ROACHE. Verification and Validation in Computational Science and Engineering. Hermosa publishers, 1998. ISBN 0-913478-08-3.
- William L. Oberkampf and Christopher J. Roy. Verification and Validation in Scientific Computing. Cambridge University Press, 2010.

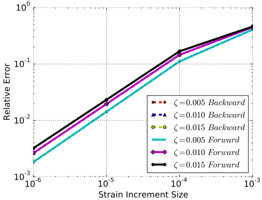
Verification

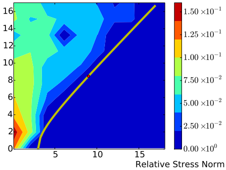
- Source code management
- Source code verification
- Constitutive integration
- Static and dynamic behavior of single phase solids
- Static and dynamic behavior of fully and partially saturated, fully coupled, porous solid-pore fluid problems
- Static and dynamic behavior of structural elements
- Static and dynamic behavior of special elements (contacts-interface/gap-frictional/dry-saturated, isolators/dissipators)
- Static and dynamic FEM solution advancement
- Seismic wave propagation problems
- FEM Model verification, hierarchy of models

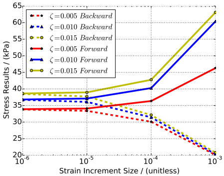
Constitutive Integration Verification

- Asymptotic regime of convergence
- Richardson extrapolation
- Grid convergence index

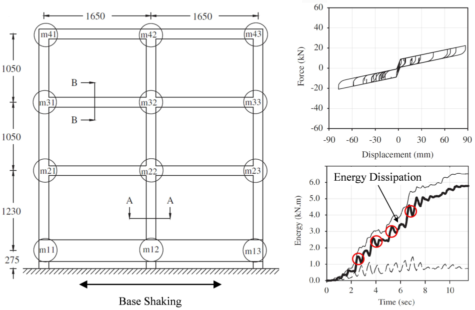








Energy Dissipation Verification: Plastic Work $\neq$ Plastic Dissipation



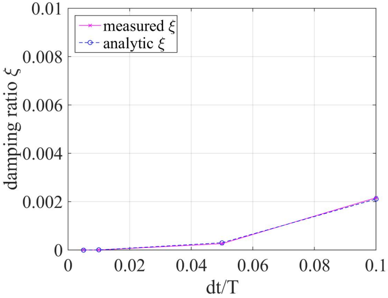
From a paper on *Soil Dynamics and Earthquake Engineering* (2011)

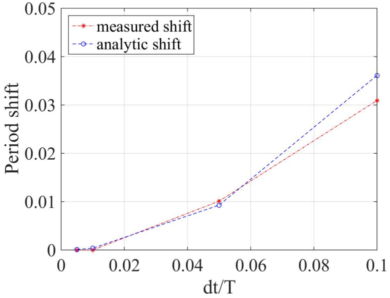
Direct violation of the second law of thermodynamics

600 papers since 1990 (!?!) repeat this error

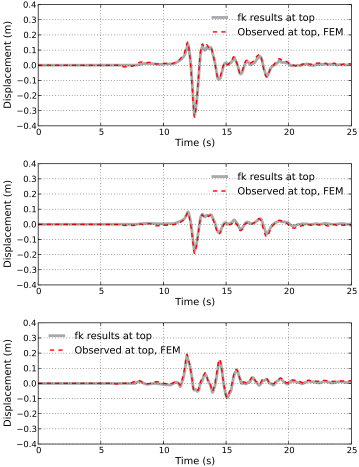
Dynamic Time Stepping Verification

Based on the amplification matrix **A**, to calculate the analytical solution of damping ratios and period shift.
Example: Hilber-Hughes-Taylor $\alpha = -0.1$

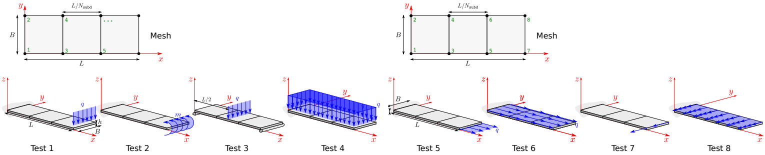




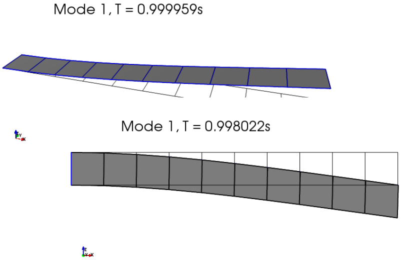
Seismic Input Verification, DRM



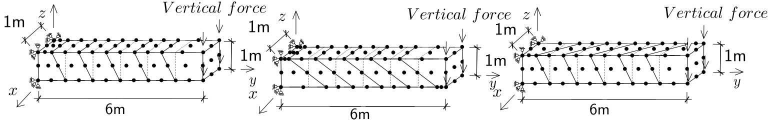
Verification: ANDES Shell



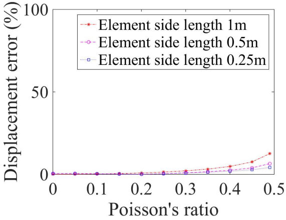
N_{subd}	u_z
2	96.2118
7	100.096
101	100.002



Verification: Irregular Solids and Poisson's Ratio

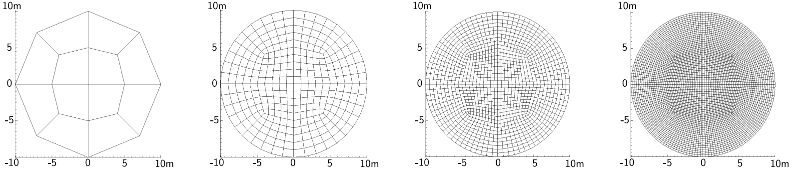


Force direction	Shape 1	Shape 2	Shape 3
Vertical (z)	0.40%	0.85%	0.60%
Transverse (y)	0.54%	3.67%	0.46%

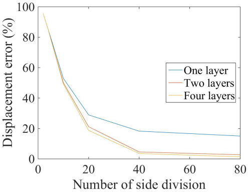


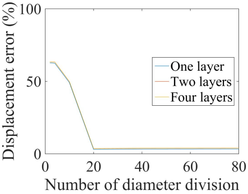
Poisson's ratio	27NodeBrick displacement	Theory displacement	Error
0.00	8.797E-04 m	8.784E-04 m	0.15%
0.05	8.801E-04 m	8.791E-04 m	0.11%
0.10	8.799E-04 m	8.799E-04 m	0.01%
0.15	8.792E-04 m	8.806E-04 m	0.16%
0.20	8.778E-04 m	8.813E-04 m	0.40%
0.25	8.758E-04 m	8.821E-04 m	0.71%
0.30	8.730E-04 m	8.828E-04 m	1.12%
0.35	8.692E-04 m	8.836E-04 m	1.63%
0.40	8.641E-04 m	8.844E-04 m	2.29%
0.45	8.567E-04 m	8.851E-04 m	3.21%
0.49	8.452E-04 m	8.857E-04 m	4.58%

Verification of Solid Shell/Plate

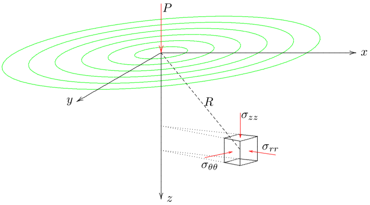


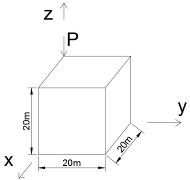
- Simply supported and clamped ends
- Timoshenko's analytic solutions

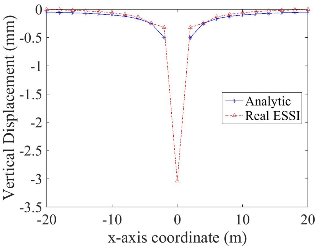


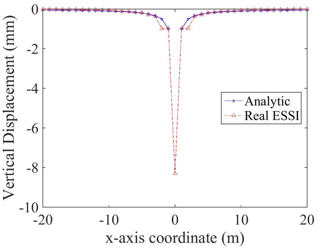


Verification of Boussinesq Problem

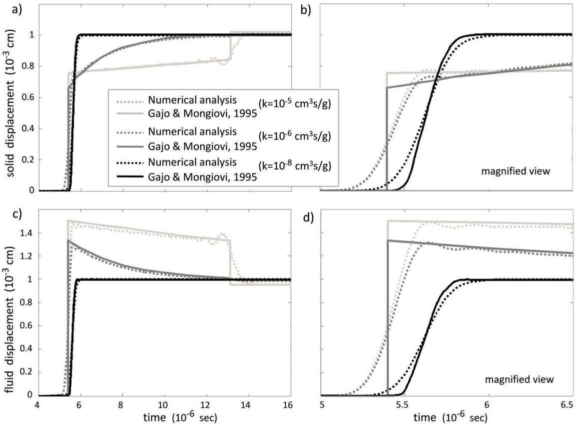




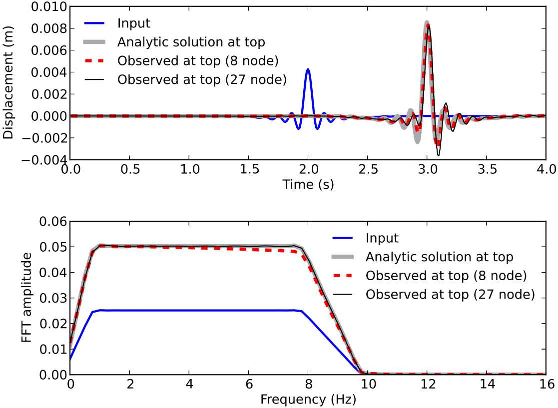




Verification for Fully Coupled Problems

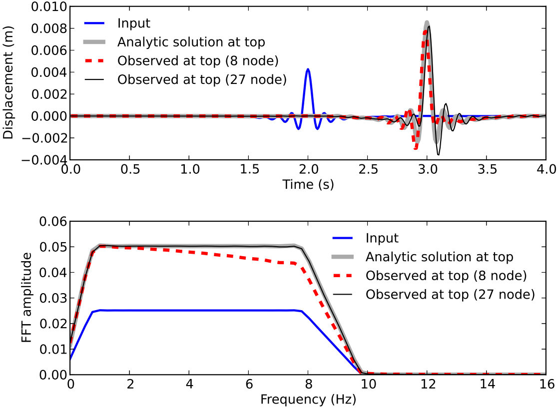


Wave Propagation, Mesh Size Effects



(Case 1, Vs = 1000 m/s, Cutoff Fq. = 8 Hz, E. Size = 10 m)

Wave Propagation, Mesh Size Effects

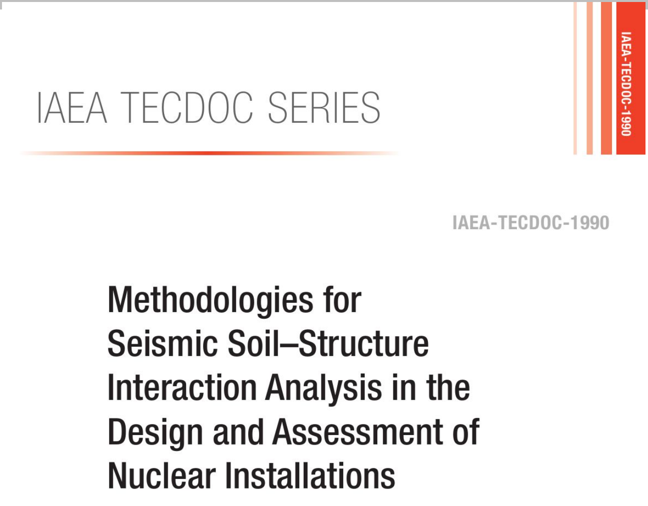


(Case 1, Vs = 1000 m/s, Cutoff Fq. = 8 Hz, E. Size = 20 m)

V & V Summary

- V&V most important for providing confidence in results
- Numerical modeling program(s) should not be used without extensive/full V&V
- V&V of FEM models is also essential
- Real ESSI Simulator has an extensive Verification database
- Validation database in development

Outline


IAEA-TECDOC-1990

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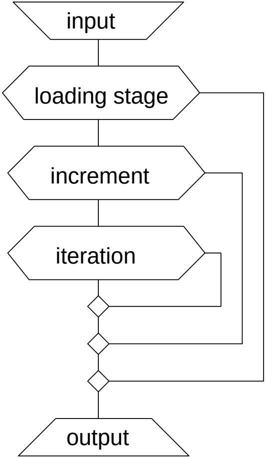
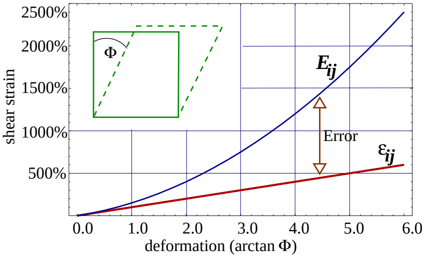
Methodologies for Seismic Soil–Structure Interaction Analysis in the Design and Assessment of Nuclear Installations

(Pecker, Johnson, Jeremić, Orbović, Altinyollar, ... (2022))

Nonlinear Modeling, Loading Stages



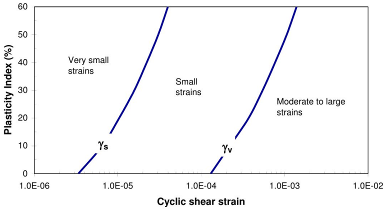
Small Deformation Theory!



$$\text{Real deformation: } E_{ij} \;=\; \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{i,k}u_{k,j}\right)$$

$$\text{Small deformation: } \epsilon_{ij} \;=\; \frac{1}{2}(u_{i,j} + u_{j,i})$$

Equivalent Linear Soil Modeling



CYCLIC SHEAR STRAIN AMPLITUDE γ		BEHAVIOUR	ELASTICITY and PLASTICIY	CYCLIC DEGRADATION in STAURATED SOILS	MODELLING
Very small	$0 \leq \gamma \leq \gamma_s$	Practically linear	Practically elastic	Non degradable	Linear elastic
Small	$\gamma_s \leq \gamma \leq \gamma_v$	Non-linear	Moderately elasto-plastic	Practically non-degradable	Viscoelastic Equivalent linear
Moderate to large	$\gamma_v \leq \gamma$	Non-linear	Elasto-plastic	Degradable	Non-linear

(Pecker, Johnson, Jeremić. Seismic Soil Structure Interaction for Design and Assessment of Nuclear Installations. ISBN-978-92-0-143021-2, UN-IAEA-TECDOC-1990, 2022)

FEM Discretization

Detailed derivation in Lecture Notes, Part I, Chapter 102

- Dynamic equilibrium: $\sigma_{ij,j} = f_i - \rho \ddot{u}_i$
- Principle of virtual displacements: $\int_V \sigma_{ij} \, \delta\epsilon_{ij} \, dV = \int_V \left(f_i^B - \rho \ddot{u}_i \right) \, \delta u_i \, dV + \int_S f_i^S \, \delta u_i \, dS$
- Displacement approximation: $u_i \approx \hat{u}_a = H_I \bar{u}_{Ia}$
- Strain: $\epsilon_{ab} \; \approx \; \hat{\epsilon}_{ab} = \frac{1}{2} \left(\left(H_{I,b} \, \bar{u}_{Ia} \right) + \left(H_{I,a} \, \bar{u}_{Ib} \right) \right)$
- Stress-strain relation: $\hat{\sigma}_{ab} = E_{abcd} \left(\hat{\epsilon}_{cd} - \epsilon^0_{cd} \right) + \sigma^0_{ab}$
- FEM discretization: $\bigcup_m \int_{V^m} \hat{\sigma}_{ab} \, \delta\hat{\epsilon}_{ab} \, dV^m = \bigcup_m \int_{V^m} \left(f_a^B - \rho \ddot{u}_a \right) \, \delta\hat{u}_a \, dV^m + \bigcup_m \int_{S^m} f_a^S \, \delta\hat{u}_a^S \, dS^m$

FEM Discretization, Tensor Form

- Final FEM equations: $M_{AacB} \; \ddot{\bar{u}}_{Bc} + K_{AacB} \; \bar{u}_{Bc} = F_{Aa}$

- Mass matrix/tensor: $M_{IacJ} = \int_{V^m} H_J \; \delta_{ac} \; \rho \; H_I \; dV^m$

- Stiffness matrix/tensor: $K_{IacJ} = \int_{V^m} H_{I,b} \; E_{abcd} \; H_{J,d} \; dV^m$

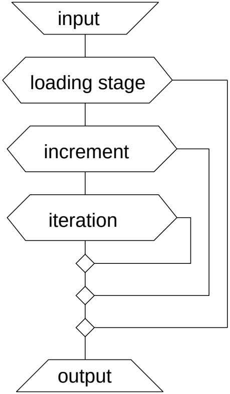
- Load vector/tensor: $F_{Ia} = \int_{V^m} f_a^B \; H_I \; dV^m + \int_{S^m} f_a^S \; H_I \; dS^m + \int_{V^m} E_{abcd} \; \epsilon_{cd}^0 \; H_{I,b} \; dV^m + \int_{V^m} \sigma_{ab}^0 \; H_{I,b} \; dV^m$

FEM Discretization, Matrix Form

- FEM matrix form: $M_{PQ}\ \ddot{\bar{u}}_P + K_{PQ}\ \bar{u}_P = F_Q$

FEM, Matrix Form, Residuals, Equilibrium

- Nonlinear FEM residuals, equilibrium:
  $R_Q = F_Q - \left(M_{PQ} \, \ddot{\bar{u}}_P + K_{PQ} \, \bar{u}_P \right)$



Introduction to Two Phase, Coupled Systems

- Based on:
  O. C. Zienkiewicz and T. Shiomi. Dynamic behaviour of saturated porous media; the generalized Biot formulation and its numerical solution. International Journal for Numerical and Analytical Methods in Geomechanics, 8:71-96, 1984.

- Full coupling of internal, pore fluid with porous solid
- Full saturation and partial, unsaturated soils
- Statics and dynamics of coupled systems

Assumptions

- 3D Geometry
- Material behavior, elastic and/or inelastic
- Mixture of pore fluid and porous solid
- Compressible pore fluid and compressible porous solid
- Small deformation

Effective Stress Principle

$$\sigma''_{ij} = \sigma_{ij} - \alpha\delta_{ij}p$$

- σ_{ij} - the total Cauchy stress in the mixture
- u_i - the displacement of the solid skeleton
- w_i - disp. of fluid phase relative to the skeleton of solid
- p - the pore fluid pressure
- $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ - strain increment of the solid phase,
- ρ, ρ_s, ρ_f - densities of mixture, solid phase and water
- $n = V_{voids}/V_{total}$ - porosity,
- $\dot{w}_{i,i}$ - rate of change of volume of fluid / volume of mixture

Governing Equations

- Equilibrium Equation of the Mixture:
  $\sigma_{ij,j} - \rho \ddot{u}_i - \rho_f [\ddot{w}_i + \underline{\dot{w}_j \dot{w}_{i,j}}] + \rho b_i = 0$
  $\rho = \frac{M_t}{V_t} = \frac{M_s + M_f}{V_t} = \frac{V_s \rho_s + V_f \rho_f}{V_t} = \frac{V_f}{V_t}\rho_f + \frac{V_t - V_f}{V_t}\rho_s = n\rho_f + (1-n)\rho_s$
- Equilibrium Equation of the Fluid
  $-p_{,i} - R_i - \rho_f \ddot{u}_i - \rho_f [\ddot{w}_i + \underline{\dot{w}_j \dot{w}_{i,j}}]/n + \rho_f b_i = 0$
  $R_i = k_{ij}^{-1} \dot{w}_j \qquad \text{or} \qquad R_i = k^{-1} \dot{w}_i$
- Flow Conservation Equation
  $\dot{w}_{i,i} + \alpha \dot{\varepsilon}_{ii} + \frac{\dot{p}}{Q} + \underline{n \frac{\dot{\rho}_f}{\rho_f} + \dot{s}_0} = 0$
  $\frac{1}{Q} \equiv \frac{n}{K_f} + \frac{\alpha - n}{K_s} \cong \frac{n}{K_f} + \frac{1-n}{K_s}$

Simplified Governing Equations

- Equilibrium Equation of the Mixture:

$$\sigma_{ij,j} - \rho \ddot{u}_i - \rho_f \ddot{w}_i + \rho b_i = 0$$

- Equilibrium Equation of the Fluid

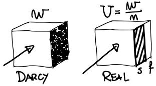
$$-p_{,i} - R_i - \rho_f \ddot{u}_i - \frac{\rho_f \ddot{w}_i}{n} + \rho_f b_i = 0$$

- Flow Conservation Equation

$$\dot{w}_{i,i} + \alpha \dot{\varepsilon}_{ii} + \frac{\dot{p}}{Q} = 0$$

Modified Governing Equations

$$U_i = u_i + \frac{w_i}{n}$$



$$\sigma''_{ij,j} - (\alpha - n)p_{,i} + (1-n)\rho_s b_i - (1-n)\rho_s \ddot{u}_i + nR_i = 0$$

$$-np_{,i} + n\rho_f b_i - n\rho_f \ddot{U}_i - nR_i = 0$$

$$-n\dot{U}_{i,i} = (\alpha - n)\dot{\varepsilon}_{ii} + \frac{1}{Q}\dot{p}$$

u_i: three solid displacement
p: pore pressure
U_i: three fluid displacement

FEM u-p-U Discretization

$$
\begin{bmatrix} M_s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & M_f \end{bmatrix}
\begin{bmatrix} \ddot{\bar{u}} \\ \ddot{\bar{p}} \\ \ddot{\bar{U}} \end{bmatrix}
+
\begin{bmatrix} C_1 & 0 & -C_2 \\ 0 & 0 & 0 \\ -C_2^T & 0 & C_3 \end{bmatrix}
\begin{bmatrix} \dot{\bar{u}} \\ \dot{\bar{p}} \\ \dot{\bar{U}} \end{bmatrix}
+
\begin{bmatrix} K^{EP} & -G_1 & 0 \\ -G_1^T & -P & -G_2^T \\ 0 & -G_2 & 0 \end{bmatrix}
\begin{bmatrix} \bar{u} \\ \bar{p} \\ \bar{U} \end{bmatrix}
=
\begin{bmatrix} \bar{f}_s \\ 0 \\ \bar{f}_f \end{bmatrix}
$$

FEM u-p-U Discretization, Index Form

$$
\begin{bmatrix} (M_s)_{KijL} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (M_f)_{KijL} \end{bmatrix}
\begin{bmatrix} \ddot{\bar{u}}_{Lj} \\ \ddot{\bar{p}}_N \\ \ddot{\bar{U}}_{Lj} \end{bmatrix}
+
\begin{bmatrix} (C_1)_{KijL} & 0 & -(C_2)_{KijL} \\ 0 & 0 & 0 \\ -(C_2)_{LjiK} & 0 & (C_3)_{KijL} \end{bmatrix}
\begin{bmatrix} \dot{\bar{u}}_{Lj} \\ \dot{\bar{p}}_N \\ \dot{\bar{U}}_{Lj} \end{bmatrix}
$$

$$
+
\begin{bmatrix} (K^{EP})_{KijL} & -(G_1)_{KiM} & 0 \\ -(G_1)_{LjM} & -P_{MN} & -(G_2)_{LjM} \\ 0 & -(G_2)_{KiL} & 0 \end{bmatrix}
\begin{bmatrix} \bar{u}_{Lj} \\ \bar{p}_M \\ \bar{U}_{Lj} \end{bmatrix}
=
\begin{bmatrix} \bar{f}^{solid}_{Ki} \\ 0 \\ \bar{f}^{fluid}_{Ki} \end{bmatrix}
$$

FEM u-p-U Matrices/Tensors

$$\boldsymbol{M_s} = (M_s)_{KijL} = \int_\Omega H_K^u (1-n)\rho_s \delta_{ij} H_L^u d\Omega \ ; \ \boldsymbol{M_f} = (M_f)_{KijL} = \int_\Omega H_K^U n \rho_f \delta_{ij} H_L^U d\Omega$$

$$\boldsymbol{C_1} = (C_1)_{KijL} = \int_\Omega H_K^u n^2 k_{ij}^{-1} H_L^u d\Omega \ ; \ \boldsymbol{C_2} = (C_2)_{KijL} = \int_\Omega H_K^u n^2 k_{ij}^{-1} H_L^U d\Omega$$

$$\boldsymbol{C_3} = (C_3)_{KijL} = \int_\Omega H_K^U n^2 k_{ij}^{-1} H_L^U d\Omega$$

$$\boldsymbol{K^{EP}} = (K^{EP})_{KijL} = \int_\Omega H_{K,m}^u D_{imjn} H_{L,n}^u d\Omega$$

$$\boldsymbol{G_1} = (G_1)_{KiM} = \int_\Omega H_{K,i}^u (\alpha - n) H_M^p d\Omega \ ; \ \boldsymbol{G_2} = (G_2)_{KiM} = \int_\Omega n H_{K,i}^U H_M^p d\Omega$$

$$\boldsymbol{P} = P_{NM} = \int_\Omega H_N^p \frac{1}{Q} H_M^p d\Omega$$

FEM u-p-U Loads

$$
\begin{aligned}
\bar{f}_{Ki}^{solid} &= (f_1^u)_{Ki} - (f_4^u)_{Ki} + (f_5^u)_{Ki} \\
\bar{f}_{Ki}^{fluid} &= -(f_1^U)_{Ki} + (f_2^U)_{Ki} \\
(f_1^u)_{Ki} &= \int_{\Gamma_t} H_K^u n_j \sigma_{ij}^{''} d\Gamma \\
(f_4^u)_{Ki} &= \int_{\Gamma_p} H_K^u (\alpha - n) n_i p d\Gamma \\
(f_5^u)_{Ki} &= \int_{\Omega} H_K^u (1 - n) \rho_s b_i d\Omega \\
(f_1^U)_{Ki} &= \int_{\Gamma_p} n H_K^U n_i p d\Gamma \\
(f_2^U)_{Ki} &= \int_{\Omega} n H_K^U \rho_f b_i d\Omega
\end{aligned}
$$

Coupled FEM Discretization, Matrix Form

- Coupled FEM, full matrix form:

$$\begin{bmatrix} M_s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & M_f \end{bmatrix} \begin{bmatrix} \ddot{\bar{u}} \\ \ddot{\bar{p}} \\ \ddot{\bar{U}} \end{bmatrix} + \begin{bmatrix} C_1 & 0 & -C_2 \\ 0 & 0 & 0 \\ -C_2^T & 0 & C_3 \end{bmatrix} \begin{bmatrix} \dot{\bar{u}} \\ \dot{\bar{p}} \\ \dot{\bar{U}} \end{bmatrix} +$$

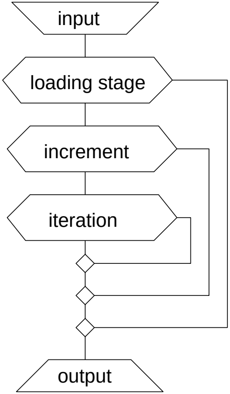
$$+ \begin{bmatrix} K^{EP} & -G_1 & 0 \\ -G_1^T & -P & -G_2^T \\ 0 & -G_2 & 0 \end{bmatrix} \begin{bmatrix} \bar{u} \\ \bar{p} \\ \bar{U} \end{bmatrix} = \begin{bmatrix} \bar{f}_s \\ 0 \\ \bar{f}_f \end{bmatrix}$$

- Coupled FEM, generalized matrix form:

$$M_{PQ}\ \ddot{\bar{u}}_P + C_{PQ}\ \dot{\bar{u}}_P + K_{PQ}\ \bar{u}_P = F_Q$$

Coupled FEM, Residuals, Equilibrium

- Nonlinear coupled FEM residuals, equilibrium:
  $R_Q =$
  $F_Q - \left(M_{PQ}\, \ddot{\bar{u}}_P + C_{PQ}\, \dot{\bar{u}}_P + K_{PQ}\, \bar{u}_P \right)$



Outline

Intro to Elasto-Plasticity

- Define elastic-plastic model: elasticity E_{ijkl}; yield function $F(\sigma_{ij}, q_*)$; plastic flow direction m_{ij}; and hardening-softening function h_*)
- Given increment in displacements Δu_i and hence increment in strain $\Delta\epsilon_{ij} = \frac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i})$
- Solve for consistency parameter $\Delta\lambda$,
- Solve for increment in plastic strain $\Delta\epsilon_{ij}^{pl} = \Delta\lambda m_{ij}$
- Solve for increment in internal variables $\Delta q_* = \Delta\lambda h_*$

Incremental Elasto–Plasticity, Components

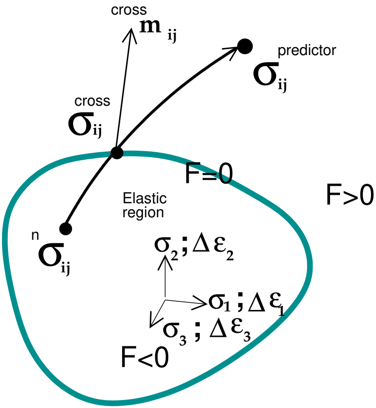
- Elasticity, $\Delta\sigma_{ij} = E_{ijkl}\Delta\epsilon^{e}_{kl}$
- Yield function $F(\sigma_{ij}, q_*) \leq 0$
- Plastic flow directions $\Delta\epsilon^{p}_{ij} = \Delta\lambda \; m_{ij}(\sigma_{ij}, q_*)$
- Hardening/softening rules $\Delta q_* = \Delta\lambda \; h_*(\sigma_{ij}, q_*)$

Elasticity

- Hyperelasticity, $\sigma_{ij} = \partial W / \partial \epsilon_{ij}$, where W is the strain energy function per unit volume
- Hypoelasticity, direct modeling of nonlinear elastic deformation
- Linear and nonlinear elastic models
- Isotropic, cross–anisotropic, full anisotropic

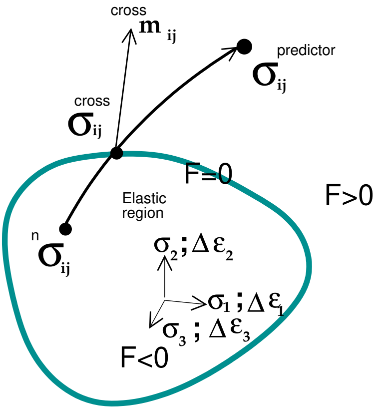
Yield Function

$$F(\sigma_{ij}, q_*) \leq 0$$



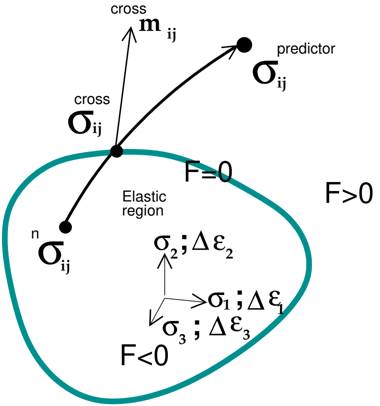
Plastic Flow Directions

$$\Delta\epsilon^{p}_{ij} = \Delta\lambda\, m_{ij}(\sigma_{ij}, q_*)$$



Hardening and Softening Rules

$$\Delta q_* = \Delta\lambda \; h_*(\sigma ij, q_*)$$



Incremental Elasto–Plasticity

- Additive decomposition of strain for small deformations
  $\Delta\epsilon_{ij} = \Delta\epsilon^{e}_{ij} + \Delta\epsilon^{p}_{ij}$

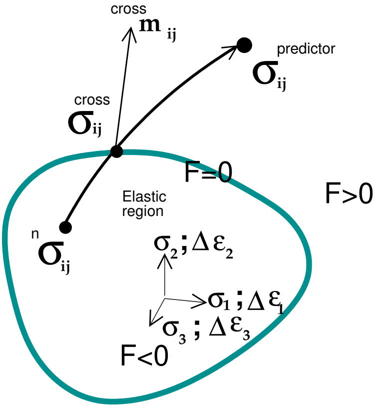
- Elastic relationship
  $\Delta\sigma_{ij} = E^{El}_{ijkl}\Delta\epsilon_{kl}$

- Elastic-Plastic relationship
  $\Delta\sigma_{ij} = E^{El-Pl}_{ijkl}\Delta\epsilon_{kl}$

Constitutive Elastic-Plastic Problem

Given increment in displacements Δu_i
and hence increment in strain $\Delta\epsilon_{ij} = \frac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i})$
determine if step is

- Elastic,
  $\Delta\sigma_{ij} = E^{El}_{ijkl}\Delta\epsilon_{kl}$
- Elastic-plastic
  $\Delta\sigma_{ij} = E^{El-Pl}_{ijkl}\Delta\epsilon_{kl}$



Constitutive Elastic Problem

For elastic step solve for

- Stiffness, E^{El}_{ijkl}, already exists from model definition
- Increment in stress, $\Delta\sigma_{ij} = E^{El}_{ijkl}\Delta\epsilon_{kl}$

Constitutive Elastic-Plastic Problem

For Elastic-Plastic step solve for

- Increment in plastic strain, $\Delta\epsilon_{ij}^{p} = \Delta\lambda\ m_{ij}(\sigma_{ij}, q_*)$
- Increment in internal variables, hardening or softening, $\Delta q_* = \Delta\lambda\ h_*(\tau_{ij}, q_*)$, that will change size, location or shape of the yield surface, $F(\sigma_{ij}, q_*)$

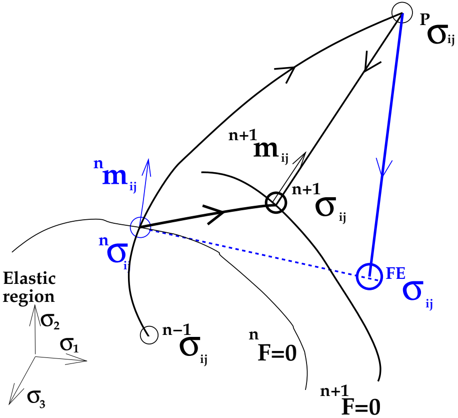
  Increment in stress, $\Delta\sigma_{ij} = E_{ijkl}^{EP}\Delta\epsilon_{kl}$

Constitutive Elastic-Plastic Problem

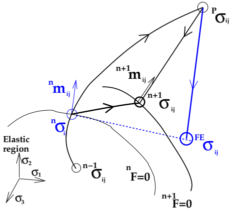
The main unknown is the consistency parameter $\Delta\lambda$ as it is needed for

- Increment in plastic strain $\Delta\epsilon_{ij}^{p} = \Delta\lambda \; m_{ij}(\sigma_{ij}, q_*)$
- Increment in internal variables, hardening or softening, $\Delta q_* = \Delta\lambda \; h_*(\tau_{ij}, q_*)$,

Explicit Integration Algorithm



Explicit Integration Algorithm

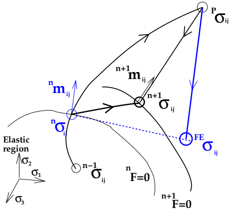


Increments

$$\Delta\sigma_{mn} = E_{mnpq}\,\Delta\epsilon_{pq} - E_{mnpq}\,\frac{{}^{n}n_{rs}\;E_{rstu}\;\Delta\epsilon_{tu}}{{}^{n}n_{ab}\;E_{abcd}\;{}^{n}m_{cd} - \xi_A h_A}\;{}^{n}m_{pq}$$

$$\Delta q_A = \left(\frac{{}^{n}n_{mn}\;E_{mnpq}\;\Delta\epsilon_{pq}}{{}^{cros}n_{mn}\;E_{mnpq}\;{}^{cros}m_{pq} - \xi_A h_A}\right) h_A$$

Explicit Integration Algorithm

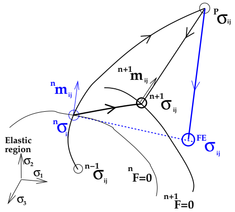


Tangent stiffness

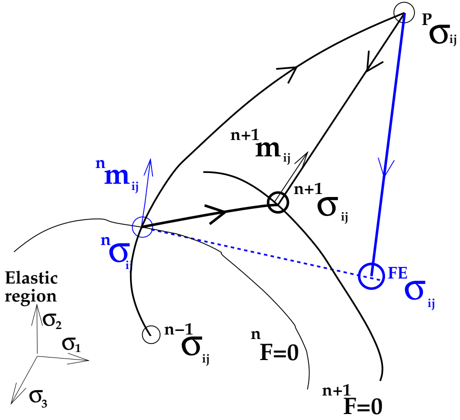
$$^{cont}E^{ep}_{pqmn} = E_{pqmn} - \frac{E_{pqkl}\,{}^{n}m_{kl}\,{}^{n}n_{ij}\,E_{ijmn}}{{}^{n}n_{ot}\,E_{otrs}\,{}^{n}m_{rs} - {}^{n}\xi_{A}\,h_{A}}$$

Explicit Integration Algorithm

- Relatively simple (first derivatives)
- Fast (single step)
- Inaccurate (accumulates error)
- Popular (most/all commercial codes)
- Works with global explicit algorithm, equilibrium: NO!

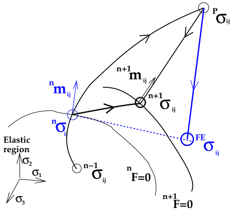


Implicit Integration Algorithm

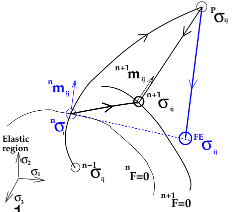


Implicit Integration Algorithm

- Elastic predictor – plastic corrector algorithm
  $${}^{n+1}\sigma_{ij} = {}^{pred}\sigma_{ij} - \Delta\lambda \; E_{ijkl} \; {}^{n+1}m_{kl}$$
- Tensor of residuals used in iterations
  $$r_{ij} = \sigma_{ij} - \left({}^{pred}\sigma_{ij} - \Delta\lambda \; E_{ijkl} \; m_{kl}\right)$$
- Iterative increments
  $$\mathrm{d}(\Delta\lambda) = \left({}^{old}f - \mathbf{n}^T \; \mathbb{C} \; {}^{old}\mathbf{r}\right) / \left(\mathbf{n}^T \; \mathbb{C} \; \mathbf{M}\right)$$
  $$\left\{ \begin{array}{c} \mathrm{d}\sigma_{mn} \\ \mathrm{d}q_B \end{array} \right\} = -\mathbb{C} \left({}^{old}\mathbf{r} + \mathrm{d}(\Delta\lambda)\mathbf{m}\right)$$
  with $\mathbf{n} = \left\{ \begin{array}{c} n_{mn} \\ \xi_B \end{array} \right\}$, $\mathbf{m} = \left\{ \begin{array}{c} E_{ijkl}m_{kl} \\ -h_A \end{array} \right\}$, ${}^{old}\mathbf{r} = \left\{ \begin{array}{c} {}^{old}\sigma_{ij} \\ {}^{old}r_A \end{array} \right\}$



Implicit Integration Algorithm



- Super-matrix $\mathbb{C}$ has different formats depending on a number and type of internal variables

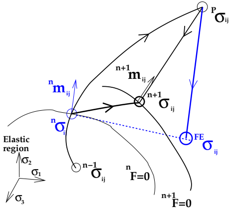
$$\mathbb{C} = \left[\; I^{s}_{ijmn} + \Delta\lambda E_{ijkl} \frac{\partial m_{kl}}{\partial \sigma_{mn}} \; \right]^{-1}$$

$$\mathbb{C} = \begin{bmatrix} I^{s}_{ijmn} + \Delta\lambda E_{ijkl} \frac{\partial m_{kl}}{\partial \sigma_{mn}} & \Delta\lambda E_{ijkl} \frac{\partial m_{kl}}{\partial q} \\ -\Delta\lambda \frac{\partial h}{\partial \sigma_{ij}} & 1 - \Delta\lambda \frac{\partial h}{\partial q} \end{bmatrix}^{-1}$$

$$\mathbb{C} =$$

$$\begin{bmatrix} I^{s}_{ijmn} + \Delta\lambda E_{ijkl} \frac{\partial m_{kl}}{\partial \sigma_{mn}} & \Delta\lambda E_{ijkl} \frac{\partial m_{kl}}{\partial z_{mn}} & \Delta\lambda E_{ijkl} \frac{\partial m_{kl}}{\partial \alpha_{mn}} \\ -\Delta\lambda \frac{\partial h^{z}}{\partial \sigma_{ij}} & I^{s}_{ijmn} - \Delta\lambda \frac{\partial h^{z}}{\partial z_{mn}} & -\Delta\lambda \frac{\partial h^{z}}{\partial \alpha_{ij}} \\ -\Delta\lambda \frac{\partial h^{\alpha}_{mn}}{\partial \sigma_{ij}} & -\Delta\lambda \frac{\partial h^{\alpha}_{mn}}{\partial z_{mn}} & I^{s}_{ijmn} - \Delta\lambda \frac{\partial h^{\alpha}_{mn}}{\partial \alpha_{ij}} \end{bmatrix}^{-1}$$

Implicit Integration Algorithm



Consistent (algorithmic) stiffness

$$\left\{ \begin{array}{c} \mathrm{d}\sigma_{ij} \\ \mathrm{d}q_A \end{array} \right\} = \left\{ \mathbb{C} - \frac{\mathbb{C}\mathbf{m}\mathbf{n}^T\mathbb{C}}{\mathbf{n}^T\,\mathbb{C}\,\mathbf{m}} \right\} \left\{ \begin{array}{c} E_{ijmn}\,\mathrm{d}\epsilon^{pred}_{mn} \\ 0 \end{array} \right\}$$

Explicit and Implicit Constitutive Integration

- Explicit, forward Euler
  - Faster elastic-plastic calculations
  - Simplest elastic-plastic integration
  - Error accumulation
  - Tangent stiffness, used for explicit, non-iterative global level
- Implicit, backward Euler
  - Slower elastic-plastic calculations
  - Sophisticated elastic-plastic integration
  - Error controlled through user defined tolerance
  - Consistent stiffness, used for implicit, Newton iterations on global level

Outline

Nonlinear FEM Equilibrium Iterations

- Global, FEM equilibrium iterations, convergence
  - Force, unbalanced, relative, abs. minimum check
  - Force, unbalanced, average
  - Force, unbalanced, absolute
  - Displacement, incremental, relative, abs. minimum check
  - Displacement, incremental, average
  - Displacement, incremental, absolute
  - Energy, incremental, relative, abs. minimum check
  - Energy, incremental, average
  - Energy, incremental, absolute

- Local, constitutive level equilibrium iterations

Available Linear/Nonlinear Programs, I

- ABAQUS, (`http://www.3ds.com`)
- ADINA, (`http://www.adina.com`)
- ALGOR/AutoDesk, (`http://www.autodesk.com`)
- ANSYS, (`http://www.ansys.com`)
- CLASSI, (`https://www.lstc.com`)
- GT STRUDL, (`https://hexagonppm.com`)
- LS-DYNA, (`http://www.lstc.com`)
- NASTRAN, (`http://www.mscsoftware.com`)
- FLUSH, (`https://www.geoengineer.org`)
- SAP2000, (`https://www.csiamerica.com`)

Available Linear/Nonlinear Programs, II

- SASSI 2010, (`http://sassi2000.net`)
- ACS SASSI, (`http://www.ghiocel-tech.com`)
- SMACS, (`https://www.osti.gov`)
- GT STRUDL, (`https://ce.gatech.edu`)
- SOFISTIK, (`http://www.sofistik.com`)
- PLAXIS, (`http://www.plaxis.nl`)
- FLAC, (`http://www.itascacg.com`)
- DYNAFLOW, (`https://blogs.princeton.edu`)
- Zsoil, (`http://www.zsoil.com`)
- FEAP, (`http://www.ce.berkeley.edu`)

Available Linear/Nonlinear Programs, III

- DEEPSOIL, (http://deepsoil.cee.illinois.edu)
- SIMQKE, (http://nisee.berkeley.edu)
- OpenSees, (http://opensees.berkeley.edu)
- Real ESSI, (http://real-essi.us)
- Code_ASTER, (http://www.code_astair.org)
- SHAKE91, (http://nisee.berkeley.edu)
- EERA and NEERA, (http://www.ce.memphis.edu)
- DESRA-2
- SUMDES
- D-MOD
- TESS

Summary

- Numerical modeling to predict and inform
- Reduction of modeling uncertainty
- Education, Training is the key!