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# CLOSED FORM FORMULAE FOR TWO INTERACTING EQUAL PARALLEL THROUGH-WALL CRACKS EMBEDDED IN AN INFINITE ELASTIC MEDIUM UNDER NORMAL TENSION

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# ABSTRACT

The paper presents closed form formulae for interaction factors relative to parallel equal through-wall cracks subjected to normal tension. Comparison with Finite Element calculations validate the results.

# **INTRODUCTION**

Crack interaction has been a widely studied topic. However exact formulae for crack interaction factors exits only for the case of coplanar through-wall cracks embedded in an infinite elastic medium subjected to normal tension. For two parallel through-wall cracks with the same boundary conditions, finite element computations, see for example Kamei (1974), Kachanov, M. (1993), Hasegawa (2006) and formulae fitted on numerical calculations, see for example Surendran (2012) have been published.

Shielding interactions which reduces the stress intensity factor are carefully analysed. Our approach is based on analytical results for the case of non-shielded cracks and on a numerical fit for shielded cracks. The approach is validated on more than one about 500 finite element calculations.

# CONFIGURATIONS

The configuration is an infinite linear elastic isotropic solid containing two embedded parallel cracks loaded by a normal traction at infinity.



Figure 1. Two equal parallel cracks (A<sub>2</sub> and B<sub>1</sub> are the inner crack tips).

Geometric shielding occurs when S<sub>x</sub> is negative.

#### FINITE ELEMENTS MODELS

A parametric 2D plane strain model has been developed within Cast3M software (Cast3M). It represents 2 interacting parallel cracks in an infinite body and submitted to a mode I loading. The reference crack size a<sub>1</sub> is equal to 1 mm in each cases. The centre part of that model is meshed with square 50 µm size quadratic elements. The Stress Intensity Factor (SIF) is determined at the 4 crack tips through the energy release rate G calculated with the  $G(\theta)$  approach developed by Destuynder (1981) and Suo (1989). The equivalent SIF

is obtained through  $K_{eq} = \sqrt{\frac{E.G}{1-\nu^2}}$  (1) where  $\nu$  is the Poisson's ratio. For some particular configurations of cracks very close to each other ( $\hat{S}_x =$ -0.9; -0.95; -0.975; -0.9917,  $\hat{S}_z = 0.1$ ; 0.2; 0.3; 0.4; 0.5), the crack interaction cannot be captured by the 50 µm square mesh (which represents 20 elements along the half crack length). A parametric study was then performed and converged to a much finer model with 120 elements along the half crack length (8µm squares elements). Specific meshes were thus used for those configurations.

# **INTERACTION OF COPLANAR THROUGH-WALL CRACKS**

The interaction coefficient  $\gamma_{12}$ , which describes the influence of crack 1 on crack 2 at the tip P, is defined by:

$$\gamma_{12} = \frac{K_{eq_{-12}}(P)}{K_{I_{-0}}}$$
(2)

where  $K_{eq_{12}}(P)$  is the SIF at the tip P of the crack 2 interacting with crack 1 and  $K_{I_{20}}(P)$  is the SIF at the tip P of the single crack 2.

Exact formulae for the SIFs have been established by several authors namely Sadowsky (1956). Erdogan (1962).

$$\gamma_{\rm A}(\lambda_{\rm x}) = \frac{\sqrt{1+\lambda_{\rm x}}}{\lambda_{\rm x}} \cdot \left[1 - \frac{1}{1+\lambda_{\rm x}} \cdot \frac{{\rm E}(\lambda_{\rm x})}{{\rm K}(\lambda_{\rm x})}\right] \tag{3}$$

$$\gamma_{\rm B}(\lambda_{\rm x}) = \frac{\sqrt{1-\lambda_{\rm x}}}{\lambda_{\rm x}} \cdot \left[\frac{1}{1-\lambda_{\rm x}} \cdot \frac{{\rm E}(\lambda_{\rm x})}{{\rm K}(\lambda_{\rm x})} - 1\right] \tag{4}$$

Where  $\lambda_x = \frac{2a}{2a+S_x}$  (5), K(k) and E(k) are the complete elliptic integrals of the first and second kind defines using the Jacobi form :

$$K(k) = \int_0^1 \frac{dt}{\sqrt{1 - t^2} \cdot \sqrt{1 - k^2 \cdot t^2}}$$
(6)

$$E(k) = \int_0^1 \frac{\sqrt{1 - k^2 \cdot t^2}}{\sqrt{1 - t^2}} \cdot dt$$
(7)

$$k = \sqrt{1 - \left(\frac{1 - \lambda_x}{1 + \lambda_x}\right)^2} = \frac{2\sqrt{\lambda_x}}{1 + \lambda_x}$$
(8)

The accuracy of S. Chapuliot's Finite Element results for  $\lambda_x \leq 0.98$  compared to the exact values is 0.005.

### INTERACTION OF PARALLEL THROUGH-WALL CRACKS

The derivation of closed form formulae for parallel cracks under normal loading requires a different approach for overlapping and no-overlapping cracks. Overlapping cracks are geometrically shielded, the interaction factor at the inner tip is less than unity whereas the SIF at the outer tips are amplified.

#### Formulae For Non Geometrically Shielded Cracks

Bueckner (1970) showed that the SIF of a plane crack loaded on its surface by the stress distribution acting on this surface in the uncracked body is given by the product of the stress distribution and a weight function g(M, P). For a mode I SIF of the crack 1 influenced by crack 2, the stress distribution is  $\sigma_z(M)$ . The weight function g(M, P) depends only on the geometry of the cracked body, P being the point on the crack front.

$$K_{12}(P) = \iint_{S} \sigma_{zz}(M) \cdot g(M, P) \cdot dS(M)$$
(9)

The application of the mean value theorem to the SIF formula gives for the through-wall crack:

$$K_{12}(P) = \sigma_{zz}(x_1, z) \cdot \iint_{S} g(M, P) \cdot dS(M) \quad x_1 \in [x_{A1}, x_{B1}]$$
(10)

 $x_1$  depends on the relative position of the two cracks and is unknown, but z is the vertical spacing between the cracks.

The SIF of the isolated crack is  $K_0(P) = \sigma_{app} \cdot \iint_S g(M, P) \cdot dS(M)$  (11) where  $\sigma_{app}$  is the remote applied stress. The stress field in an elastic infinite through-wall cracked body loaded by a uniform normal stress distribution has been derived by Westergaard (1939). The stresses are obtained by the following complex function and its derivatives:

$$Z_{I} = \sigma_{app} \cdot \left(\frac{z_{c}}{\sqrt{z_{c}^{2} - a^{2}}} - 1\right)$$
(12)

where  $z_c = x + i.z$  is the complex variable and a the half crack length.

The normal and the shear stresses, in terms of polar coordinates r,  $\theta$  are given by:

$$\sigma_{zz} = \sigma_{app} \cdot \frac{r}{\sqrt{r_1 \cdot r_2}} \cdot \left[ \cos\left(\theta - \frac{\theta_1 + \theta_2}{2}\right) + \frac{a^2}{r_1 \cdot r_2} \cdot \sin\theta \cdot \sin\frac{3}{2} \cdot (\theta_1 + \theta_2) \right]$$
(13)

$$\sigma_{xz} = \sigma_{app} \cdot \frac{r}{\sqrt{r_1 \cdot r_2}} \cdot \frac{a^2}{r_1 \cdot r_2} \cdot \sin \theta \cdot \cos \frac{3}{2} \cdot (\theta_1 + \theta_2)$$
(14)

Where  $r_1^2 = r^2 + a^2 - 2a \cdot r \cdot \cos \theta$  and  $r_2^2 = r^2 + a^2 + 2a \cdot r \cdot \cos \theta$ 

Two Westergaard functions are thus obtained:

$$F_{W_{zz}}\left(\frac{x_1}{a}, \frac{z}{a}\right) \equiv \frac{\sigma_{zz}}{\sigma_{app}}$$
 and  $F_{W_{zz}}\left(\frac{x_1}{a}, \frac{z}{a}\right) \equiv \frac{x}{\sigma_{app}}$  (15)

Aiming to take into account the shear induced by the isolated crack, we define:

$$F_{W_eq}\left(\frac{x_1}{a}, \frac{z}{a}\right) = \sqrt{F_{W_zz}^2 + F_{W_zz}^2}$$
(16)

Therefore, the interaction factor is simply expressed by that we call the Westergaard function FW:

$$\gamma_{12}(P) = F_{W_eq}\left(\frac{x_1}{a}, \frac{z}{a}\right)$$
(17)

For z = 0, the interaction factor is expressed by the exact formulae (3, 4), then x1 is obtained by the non-linear equation:

$$F_{W_eq}\left(\frac{x_1}{a},0\right) = \gamma_A \text{ or } \gamma_B$$
(18)

Eventually the mode I interaction factor for the parallel crack configuration is exactly determined.

#### Analysis of very close parallel cracks

The interaction factors of overlapped cracks located at very close spacings are difficult to compute accurately. A reliable numerical result, based on crack representation by distributions of dislocations, has been provided by Kamei and Yokobori (1974). The approximate method developed by Kachanov (1993) deviates from this solution when the spacing becomes smaller than  $S_z/(2a) = 0.5$ .

Gorbatikh (2007) proposes a method to predict stress intensity factors (SIFs) of strongly interacting cracks at spacings that are substantially smaller than crack lengths. The double parallel cracks with the same length, when they are very close, they strongly shield each other. Figure 3 shows that Kamei results are in agreement with Gorbatikh's asymptotic analysis. Using these two results we have developed the following fitted equations:

$$\gamma_{S_{\perp}I} = 1 - 0.435 * \exp(-1.015 * \hat{S}_z^{0.727})$$
<sup>(19)</sup>

$$\gamma_{S_{II}} = 0.335 * \exp(-1.675 * \hat{S}_z^{0.751})$$
<sup>(20)</sup>



 $\gamma_{S\_eq} = \sqrt{\gamma_{S\_I}^2 + \gamma_{S\_II}^2}$ (21)

Figure 1. Interaction factors for stacked equal through-wall cracks.

# Finite Element Results

S. Chapuliot (2018) completed several Finite Element computations for interaction modelling. Further computations have been conducted leading to a large set of interaction factors: about 36 values of  $\hat{S}_x$  are selected and  $\hat{S}_z$  varies from 0,1 to 2 by steps of 0,1. Including additional accuracy tests, the whole set contains 890 computations.

Figure 2 illustrates the variations of the interaction factor as a function of the relative horizontal and vertical spacings obtained by Finite Element and by the formula for the parallel cracks (square black marks).



Figure 2. Finite Element computed interaction factors.

These results show that the FE computations converge toward the analytical values for stacked cracks.

The tightness of the FE grid allows to determine the shielding limit. Shielding occurs when the following inequality is verified:

$$\gamma_{12\_Int} \leq \gamma_{12\_Ext} \tag{22}$$

The shielding limit. has been computed by several authors: Kamei's study gives results almost on the top of the estimation based on our FE computations. D. Zhu (2021) obtains a curve of similar shape but stiffer. This proves that shielding cannot be always predicted by geometrical considerations, but depends on the material behaviour law.



Figure 3. Shielding limits computed by Finite Element, dislocations technique and analytical formulae.

The shielding limit. may also be computed using the  $\gamma_{eq}$  formulae, leading to an analytical limit (Shielding limit\_Eq), which is very close to the FE as shown on figure 3. The differences observed at low spacings is due to the influence of Mode II, which is partially taken into account in our approach.

The FE computed shielding limit is accurately described by the following set of equations:

$$\begin{cases} \hat{S}_{xE} = \hat{S}_{x0} - t \cdot \hat{S}_{zE} \\ \hat{S}_{zE} = \hat{S}_{z0} - 3a \cdot \frac{t}{1+t^3} \end{cases} \text{ where } a = 0.4, \, \hat{S}_{x0} = -0.136, \, \hat{S}_{z0} = 0.215 \tag{20}$$

The equivalent shielding limit is accurately described by the following equation:

$$\hat{S}_{x\_SEq} = -0.379 \cdot \hat{S}_{z} - 1.6268 \cdot \hat{S}_{z}^{2} + 2.4547 \cdot \hat{S}_{z}^{1.8323}$$
(21)

#### Formulae For Elastically Shielded Cracks

Equations (13) to (17) cannot be used for the prediction of the interaction factor of overlapped cracks. The variations of the interaction factor for negative values of  $\hat{S}_x$  being highly non-linear, we use the equation 18 to determine the  $x_R = \frac{x_1}{a}$  value for each spacing  $\hat{S}_z$  from the FE results. The variations of  $x_R$  are illustrated in Figure 4 for the inner tip. These variations are smoother than those of interaction factors, namely for the shielded cracks. The  $x_R$  values are then fitted for the shield cracks. For outer tips, the equivalent shielding limit has been slightly modified for configurations corresponding to  $\hat{S}_z \leq 0.7$ .

$$\hat{S}_{x\_SEq} = \frac{0.2145}{0.7} \cdot \hat{S}_z$$
 (22)

Fitting formulae are given in the appendix.



Figure 4. Variations of the X<sub>R</sub> values as a function of the spacing parameters  $\hat{S}_x$  and  $\hat{S}_z$ .

# VALIDATION OF THE FORMULAE

Figure 5 compares the results obtained using our analytical formulae (marks) to the Finite Element results (solid lines).



Figure 5. Comparison of FE computed and analytic interaction factors for two equal parallel cracks.

For non-shielded cracks, the agreement between EF and predicted results is excellent. For shielded cracks, the non-linear fits give very good predictions except at the outer tip for very close cracks ( $S_z/(2a) = 0.1$ ).

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# CONCLUSION

Analytical formulae have been established for interaction factors at the inner and outer tips of two parallel cracks and validated by comparison to FE results. Shielding is shown as depending on the material behaviour law and cannot be reduced to geometrical characteristics. Closed form formulae have also been derived for two stacked parallel cracks.

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# APPENDIX Fitting formulae for interaction factors of shielded cracks

The given spacings are  $\hat{S}_{x0}$  and  $\hat{S}_{z0}$ .

### **INNER TIP**

 $\hat{S}_z \leq 0.6$ 

 $A = -1.0033 \cdot S_{z0}^{3} + 0.9125 \cdot S_{z0}^{2} + 0.3602 \cdot \hat{S}_{z0} + 0.9795$   $B = -6.6815 \cdot S_{z0}^{3} + 8.6734 \cdot S_{z0}^{2} - 2.45 \cdot \hat{S}_{z0} + 1.2583$   $C = -41.714 \cdot S_{z0}^{3} + 67.541 \cdot S_{z0}^{2} - 35.532 \cdot \hat{S}_{z0} + 5.3758$   $D = -71.356 \cdot S_{z0}^{3} + 120.4 \cdot S_{z0}^{2} - 68.799 \cdot \hat{S}_{z0} + 11.094$   $E = -37.332 \cdot S_{z0}^{3} + 63.837 \cdot S_{z0}^{2} - 37.645 \cdot \hat{S}_{z0} + 6.976$   $\hat{S}_{z} > 0.6$   $A = -0.1078 \cdot S_{z0}^{3} + 0.2237 \cdot S_{z0}^{2} + 0.1073 \cdot \hat{S}_{z0} + 1.1797$   $B - 0.0849 \cdot S_{z0}^{3} + 0.1075 \cdot S_{z0}^{2} + 0.1392 \cdot \hat{S}_{z0} + 1.4382$   $C = -0.4952 \cdot S_{z0}^{3} + 1.895 \cdot S_{z0}^{2} - 1.943 \cdot \hat{S}_{z0} + 0.5988$   $D = -0.4746 \cdot S_{z0}^{3} + 2.0867 \cdot S_{z0}^{2} - 2.36 \cdot \hat{S}_{z0} - 0.234$ E = 0

#### **OUTER TIP**

$$\begin{split} \hat{S}_{z} &\leq 0.5 \\ A &= -1.9 \cdot S_{z0}^{3} + 3.93 \cdot S_{z0}^{2} - 0.948 \cdot \hat{S}_{z0} + 0.9643 \\ B &= -12.417 \cdot S_{z0}^{3} + 13.482 \cdot S_{z0}^{2} - 4.4496 \cdot \hat{S}_{z0} + 0.066 \\ C &= -85342 \cdot S_{z0}^{3} + 84.246 \cdot S_{z0}^{2} - 23.547 \cdot \hat{S}_{z0} + 0.424 \\ D &= -145.43 \cdot S_{z0}^{3} + 150.95 \cdot S_{z0}^{2} - 41.021 \cdot \hat{S}_{z0} + 0.8975 \\ E &= -72.783 \cdot S_{z0}^{3} + 79.614 \cdot S_{z0}^{2} - 22.571 \cdot \hat{S}_{z0} + 0.5418 \\ \hat{S}_{z} &> 0.5 \\ A &= 2.28358 - 1.5837 \cdot S_{z0}^{0.06627} \cdot Exp(-1.9225 \cdot S_{z0}^{2.24237}) \end{split}$$

 $B = 1.25447 - 5.3899 \cdot S_{z0}^{0.95834} \cdot Exp(-2.0908 \cdot S_{z0}^{2.0809})$   $C = -7142.35 \cdot S_{z0}^{5.3429} \cdot Exp(-8.5377 \cdot S_{z0}^{0.7747})$   $D = 0.01317 - 3.1623 \cdot S_{z0}^{3.1879} \cdot Exp(-1.3178 \cdot S_{z0}^{3.2597})$  E = 0

Eventually

$$x_1 = A + B \cdot \hat{S}_{x0} + C \cdot \hat{S}_{x0}^2 + D \cdot \hat{S}_{x0}^3 + E \cdot \hat{S}_{x0}^4$$