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Division I

Stochastic modelling of cracks' spacing in RC structures in the presence of size effects

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INTRODUCTION

In current engineering practice, the control of the cracking state is aimed at using a set of design guidelines (Eurocode 2; Model Code 1990; Model Code 2010 and others) to limit the stress state, the crack width and the deflection level in reinforced concrete elements. It all comes eventually to defining an accurate reinforcement ratio ρ to keep a maximal spacing value $s_{r,max}$ smaller than a given criterion for a given Limit State s_r^{LS}. So, the challenge consists of predicting for each concrete type what would be its associated maximal spacing representing a stabilized cracking state and allowing the quantification of the maximal crack opening based on the strain gap between steel and concrete.

The maximal spacing value is a random quantity; meaning that one gets a maximal value given a certain probability of exceedance p_f or a confidence interval. The uniqueness of such value is only obtained when the probability threshold (on the high side of the mean value) is fixed. Accordingly, for an accurate measurement of the maximal spacing for a given concrete type, a large set of tests should be achieved to explore tail distributions. As this remains hard to achieve, usual practice consists of considering a limited number of experimental tests to quantify the mean spacing value $s_{r,m}$ and associate an amplification factor $\lambda = s_{r,max}/s_{r,m}$ (from 1.5 in Code Mode 1990 to 1.7 in Eurocode 2 and Model Code 2010) representing the tail distribution to the best of our knowledge. However, one can note that such empirical approaches – or at best semi-empirical – do not allow a reliable quantification of the risks associated with the control of cracking in structures. Indeed, they remain intrinsically dependent on the selected experimental design plan, which explains the observed discrepancies among existing regulatory codes, Lapi, M., et al. 2018..

To overcome the limitations related to the experimental characterization of tail distributions of cracks' spacing values, numerical analyses are usually of interest. In the literature, one can find several approaches to deal with cracks' spacing quantification. Early works on the topic involve the demonstration of the experimentally observed dependence of the mean spacing $s_{r,m}$ on the ratio φ/ρ of steel diameter φ to the effective reinforcement ratio ρ (bond theory in Watstein & Parsons (1943)) and its dependence on the cover distance c (no bond theory in Broms (1965)). In terms of recent advancement on the matter, Beeby (2004) proposes a model based on a steel to concrete transfer length and demonstrates theoretically the linear relationship between the mean spacing value $s_{r,m}$ and the product of ratios $(\varphi/\rho)(R_t/\tau_{max})$ with φ the bar diameter, ρ the reinforcement ratio, R_t the tensile strength and τ_{max} the maximal steel-concrete bond. The same formulation is retained in Model Code 2010. Yet, the constants of the linear equation remain defined based on empirical fitting (best-fit equation based on statistical analysis of experimental data). Later on, Beeby & Scott (2004) extended the model in Beeby (2004) to a probabilistic framework using a normal and uncorrelated spatial random field associated with the tensile strength R_t over a discretized beam's length. Then, through several runs, authors compute iteratively the positions of cracks and define the probability density function of spacing values. Though it is an interesting approach, authors did not consider any size effects in their developments nor justify the nature of the used random field. Indeed, the tensile strength follows rather a Weibull distribution than a normal one, with different tail behaviors as proposed by Bažant & Le (2017). Also, Beeby (2004), Beeby & Scott (2004) neglected the spatial correlation of R_t which is a strong hypothesis given experimental evidence on the matter – see Vořechovský (2008).. Such spatial correlation is directly linked to the material intrinsic heterogeneity (voids and aggregates spatial distribution).

Hence, the main purpose of this paper is to suggest a new theoretical and practical methodology allowing the accurate and physical prediction of cracks' spacing distribution in reinforced concrete structures. The foreseen improvements concern (a) the introduction of size effects in the physical formulation of the model (b) the introduction of the spatial distribution of the tensile strength in a realistic way (spatially correlated Weibull distribution) and (c) the probabilistic coupling using non-intrusive techniques.

PHYSICAL MODEL FOR CRACKING IN RC BEAMS

The proposed model hereafter is based on the same basic principle as the one used in Beeby (2004) using a representative reinforced concrete beam with a length l, a concrete section of A_c, a steel section of A_s (associated to a diameter φ) and a bond-slip law subjected to a normal and axial load N.

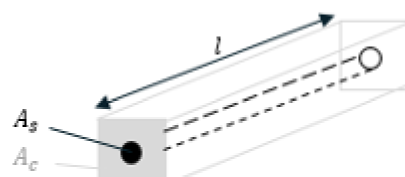

Figure 1. Visualization of the beam element between two boundaries (c and s stand for concrete and steel materials respectively).

Given the hypotheses of perfectly brittle concrete behavior, linear elastic behavior of steel prior to crack stabilization and a linear bond-slip law ($\tau_b(s) = a\ s$), one can easily solve the bond-slip differential equation. In terms of stress distribution in concrete between two cracks (at positions $-\frac{l}{2}$ and $\frac{l}{2}$ with the element's origin at mid-distance), this writes:

$$\sigma_c(x) = \frac{N\pi\varphi}{A_c A_s E_s}\left(1 - \frac{\cosh(\sqrt{a\chi}\,x)}{\cosh\left(\sqrt{a\chi}\frac{l}{2}\right)}\right) \tag{1}$$

One obtains a new crack within the element length l when the stress level σ_c at a given position reaches the material tensile strength at the same point: $\sigma_c = R_t$. In the absence of spatial variation, this happens at mid length of the free edges. To be more physically representative, two main improvements are considered:

(a) Definition of the tensile strength in concrete using an adapted Size Effect Law. In this work, we are only interested in Type I size effect (denoted also energetic-statistical size effect in Bažant & Chen (1997)) in the absence of deep notches or a large traction-free cracks in the structural volume (this is in line with the multi cracking behaviour observed in the case of 1D-tensionned beams). The size effect law generally involves an effective volume V at the structural element scale, a reference volume V_0 at which the measurement of the tensile strength is achieved $R_{t,0}$ and its associated Weibull modulus m. Volumes V_0 and V are estimated using the criterion defined in the WL² model in Sellier & Millard (2014).

$$R_t(V) = R_{t,0} \left(\frac{V}{V_0}\right)^{-\frac{1}{m}} \tag{2}$$

(b) Definition of a spatially correlated Weibull random field associated to the tensile strength using an exponential quadratic correlation function a given fluctuation length l_{flu} – see Baroth et al. (2011), Loève (1960), Nataf (1962).

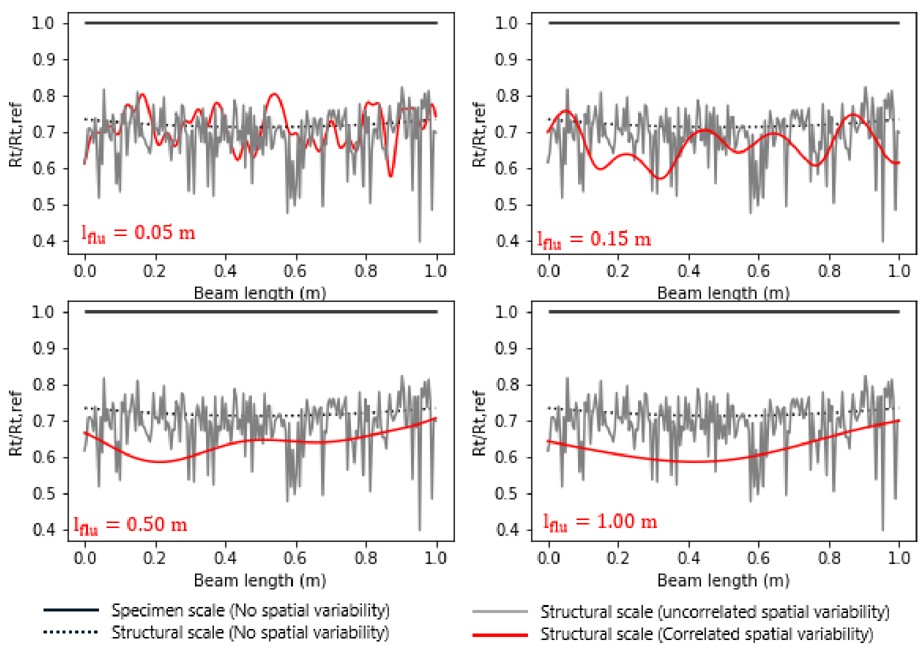

Figure 2. Effect of spatial correlation on the Weibull random field realizations.

Calculations are performed according to a step by step process:

(a) Define a representative beam section and the material properties
(b) Discretize the beam into finite elements (the number of elements should be sufficient to describe objectively the spatial correlation and the number of cracks)
(c) Evaluate size effects on the tensile strength in the beam
(d) Generate several realizations of the random field associated to the tensile strength
(e) For each realization, use an iterative process to increase the axial load activating one crack at a time (where the stress in concrete reaches the tensile strength) until the stabilized stage is reached.
(f) Consider all cracks' position to post-process the probability density function and achieve reliability analysis if necessary.

One should note that for a number of simulations $\mathrm{N_{simu}}$, one can only quantify tail probabilities higher than $p_f = 10^{2-\log_{10} N_{simu}}$).

EXPERIMENTAL VALIDATION

In the experimental work of Farra & Jacoud (1993), several reinforced concrete beams are subjected to tensile loads for cracks' spacing identification at the stabilized state. The cross section of these beams is 10 cm x 10 cm. The length is 1.15 m. Three reinforcement ratios (0.79%, 1.56%, and 3.24%) are considered. Seven concrete types with compressive strength between 29.9 MPa and 55.4 MPa are explored. The reference tensile strength values are measured using cylindrical specimen of 16 cm x 32 cm under direct tensile loads. Finally, for each specimen, three tests are performed. This is rather limited for an objective experimental probabilistic quantification. However, the observed results can still be considered for the model validation; especially in terms of physical tendencies and the quantitative analysis of the mean spacing values.

Numerical simulations are limited herein to $\mathrm{N_{simu}} = 1000$ realizations of the Weibull random field for each beam.

The obtained results summarized in Table 1 show that the fully predictive model shows a satisfactory estimation of the mean spacing compared to other empirical models fitted empirically to the data base. Nevertheless, for those empirical models, such accuracy is not guaranteed if other data are considered. It is important to underline that size effects are not that pronounced at the scale of Farra & Jacoud beams ($\mathrm{R_t(V_{eff})/R_{t,0}} \approx 0.93$) which explains why empirical models using the tensile strength at the specimen scale still work accurately. As for the randomness of the spacing values for each beam, the Model Codes applied within a probabilistic framework underestimates the variation of the spacing values. In part, this is directly due to the lack of spatial variation in the Model Code formula. On the other hand, the newly proposed model shows higher variation that depends on the considered concrete type and the measured randomness of the tensile strength.

Table 1: Mean spacing values and their coefficients of variation: experimental vs. numerical.

Beam reference	Experimental	Models			
		New model (Fully predictive)		Model Code (Empirical)	
	Mean value (mm)	Mean value (mm)	CoV (%)	Mean value (mm)	CoV (%)
N10-10	237	213	21	214	3
N10-14	173	146	19	174	2.6
N10-20	129	118	19	138	2.2
N20-10	222	238	22	221	3.6
N20-14	146	156	20	179	3.2
N20-20	141	118	18	142	2.7
N30-10	208	233	23	216	3.6
N30-14	180	150	20	175	3.1
N30-20	142	114	17	140	2.6
N40-10	193	239	22	215	2.4
N40-14	180	147	20	175	2.1
N40-20	142	110	16	139	1.8
N12-10	238	200	24	191	4.8
N12-14	165	129	21	158	4.1
N12-20	153	104	12	128	3.4
N22-10	192	197	25	187	5.2
N22-14	168	126	21	154	4.4
N22-20	131	102	11	126	3.6
N32-10	236	194	24	185	3.7
N32-14	152	120	20	153	3.1
N32-20	137	100	8	125	2.6
N42-10	208	213	25	191	4.3
N42-14	160	126	20	158	3.7
N42-20	130	101	9	128	3.0

Also, given the probabilistic framework of the newly suggested method, one can access to the cumulative distribution functions of the cracks' spacing values. Based on a selected probability of failure, one can also define a maximal spacing value not to be exceeded for the crack control within the design phase. In practice, one selects a 1-α quantile which represents the risk of having values at tail distribution. For the spacing values, the interest is geared towards the high range of values (as it increases the crack opening values). So, by giving the value of α, one defines a probability of not exceeding a maximal value of which the probability is 1-α/2. One should note that the value of α varies depending on the considered application and the operational context.

Finally, compared to other methods, the suggested model offers a rigorous theoretical framework to (a) compute the crack spacing values in a fully predictive way based on measurements achieved at the specimen scale (empirical fitting is absent at the beam scale) (b) predict representative variation of the spacing values accounting for size effects and for the spatial randomness of the tensile strength (c) predict the full cumulative distribution function (or probability distribution function) to deduce the maximal spacing value not to be exceeded for a given probability of failure.

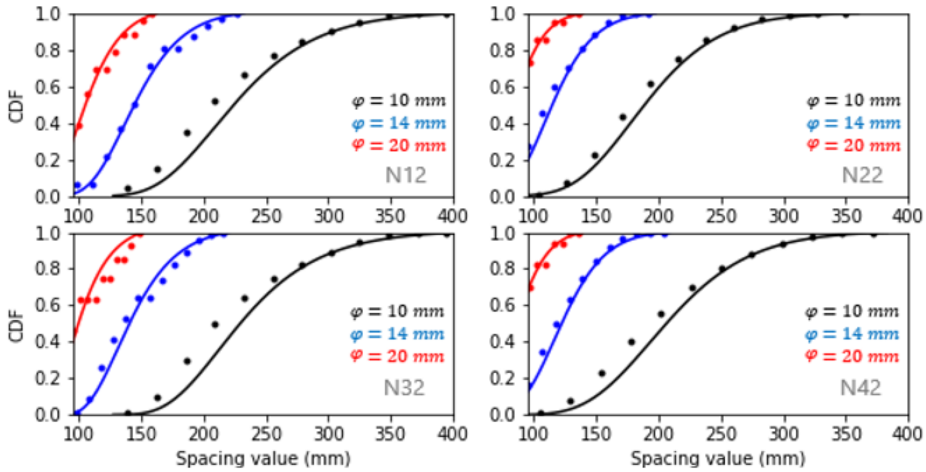

Figure 3. Cumulative distribution functions (CDFs) of the spacing values.

A NEW FORMULATION OF THE MEAN AND MAXIMAL CRACKS' SPACING VALUES

Theoretical formulation

The foreseen function shape is inspired from the Model Code formulae with the introduction of size effects parameters (based on the uses size effect law):

$$s_{rm} = A_1 * c + A_2 \left(\frac{R_{t,0}}{\tau_{max}}\right)\left(\frac{V}{V_0}\right)^{-\frac{1}{m}}\left(\frac{\varphi}{\rho}\right) \tag{3}$$

with c the cover distance, $R_{t,0}$ the tensile strength measured at the reference volume V_0, τ_{max} the maximal bond stress.

As for the maximal spacing value, defined using the ratio $\lambda = s_{r,m}/s_{r,max}$ in current regulatory codes, the proposed model allows the identification of the following relationship (associated to a Lognormal distribution of spacing values):

$$\lambda(\alpha) = \frac{\exp(\sqrt{2}\,\sigma\, k_\alpha + \mu)}{s_{rm}} \tag{4}$$

with $k_\alpha = 1.1\, e^{-1{,}92\,\alpha}$, $\mu = \ln\left(s_{rm}/\sqrt{1 + CoV_{sr}/s_{rm}}\right)$, $\sigma = \sqrt{\ln(1 + CoV_{sr}/s_{rm})}$ and CoV_{sr} the coefficient of variation associated to the spacing values.

Identification of constants

In this part, it is proposed to investigate numerically the size effects on the computed mean and coefficient of variation of the spacing values. To do so, the same reinforcement ratios defined by Farra & Jacoud (1993) are retained and extended to the following numerical design plan for the following beam sections: 0.1 x 0.1 (reference beam), 0.2 x 0.2, 0.5 x 0.5, 0.8 x 0.8, 1.2 x 1.2. One should note that the last sections are rarely encountered in standard engineering applications and are merely considered to explore the full size effect law (for small and large volumes).

Based on the achieved numerical calculations, the unknown parameters verify A_1 =1.36, A_2 =0.65, $m = 0.2 + 1.2/CoV_{R_t}$ and $CoV_{sr} = 0.279\, CoV_{R_t} + 0.19$ with $\mathrm{CoV}_{\mathrm{R_t}}$ the measured coefficient of variation of the tensile strength (usually around 10% at the specimen scale).

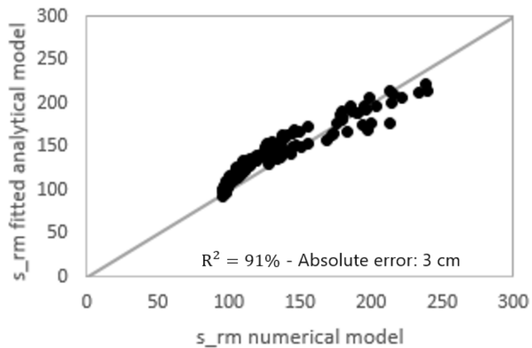

Figure 4. Comparative analysis of numerical and fitted mean spacing values.

Impact on the estimated mean and maximal cracks spacing values

Eventually, as one defines a given probability of failure $p_f = 1 - \alpha/2$, one gets a corresponding ratio λ and a maximal spacing threshold for design purposes. As one can see, the values of λ are not constant and not forcibly equal to 1.5 and 1.7 as provided in Eurocode 2 and Model Codes; at least not for low values of α. In other words, the values in regulatory codes refer to high values of $20\% \leq \alpha \leq 30\%$ which refers to probabilities of failure around $0.85 \leq p_f \leq 0.9$ (and not 0.95 as usually used in engineering applications).

In fine, by using the set of equations here above, one can easily deduce the mean and maximal spacing values of a given beam characterized by its size, stress distribution (tensile vs. bending), ratio φ/ρ, variation of the tensile strength $CoV_{R_t}$and probability of exceedance $p_f = 1 - \alpha/2$

CONCLUSION

The main aim of this research paper is to propose a probabilistic formulation of the crack's spacing values for axially reinforced members subjected to pure tensile loads. Such effort is of interest to enhance the predictive crack control in terms of mean and maximal crack spacing values and its associated crack openings. To reach the foreseen goal, the following theoretical advancements are achieved (a) considering an analytical solution for the bond-slip differential equation based on a linear bond-slip law (b) considering an energetic-statistical size effect law relating the tensile strength property to the effective structural volume (c) considering a spatially correlated Weibull random field associated to the tensile strength property. By the use of Monte Carlo Methods (which involves in this work the generation of several random fields), the stochastic model leads to the identification of a probability density function of the spacing values. From this distribution, one gets a mean estimate, a coefficient of variation and, given a certain probability threshold, the maximal crack spacing not to be exceeded.

Eventually, by exploring an extended numerical design plan, a set of equations is suggested for engineering applications. These expressions relate the mean spacing value to the ratio φ/ρ and the ratio $R_{t,0}/\tau_{\max}$ as already suggested in the Model Code formulae. In this work, it is recommended to include also a term proportional to the volume of the beam to the power (-1/m) to account for size effects on the tensile strength. As for the maximal spacing value, a theoretical approach is proposed based on the hypothesis of a Lognormal distribution of spacing values and of a coefficient of variation strongly correlated to the coefficient of variation of the tensile strength. Both of these hypotheses are explored in the present work by numerical means.

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